UIL Calculator Applications Contest Manual

Professor David L. Bourell, Ph.D., P.E.
Temple Foundation Professor
Department of Mechanical Engineering
The University of Texas at Austin

UIL Calculator Applications Director

Preface

This manual is an incremental revision of the 2010 manual. Very few items have changed. The main changes to the contest deal with the old study list and Page 7 problem location. The two are related. The study list and the relocation of problems on Page 7 are attempts to encourage contestants to attempt all the problems on the last page. The study list was a group of stated and geometry problems from which the Page 7 problems were drawn. This approach was largely unsuccessful, so the study list was eventually abandoned, and the Page 7 numerical problems were moved to the bottom of the page from the top. This makes the stated and geometry problems on Page 7 worth the same as other skipped problems on the test, assuming the contestant works the numerical problems on Page 7.

Another major change in the contest deals with clearing of calculators. Currently, calculator storage, both numerical and program space, is not cleared prior to the start of the contest. This both encourages contestants to use the space wisely and to prepare programs that may help with problem solving, and it is a great relief to contest directors who do not need to feel responsible for either clearing calculators or disqualifying contestants who violate the rules.

A major development in the calculator world over the last 10-15 years is the emergence of the handheld computer. The ti Nspire and hp Prime calculators are examples. I have conducted an informal analysis of a "powerhouse" school that changed from ti graphing calculators to Nspires, and I found no significant change in the scores over the several years of that conversion. An analysis of the calculators used by winners at the State Meet also showed no particular benefit from using the handheld computers. This is reasonable given the nature of the problems on the test.

The day is approaching but has not yet arrived, when calculators will be caught up in the Internet of Things. That is, one might expect that eventually, calculators, like cell phones, will be internet connected with the world suddenly accessible. One feature of this may well be the ability of calculators to scan equations and immediately solve them. When this day arrives, the Calculator Applications Contest will need to be reevaluated, particularly with respect to the numerical problems (or number crunchers). Historically, the number crunchers have served as a way for beginning students to find a connection to the contest. When one can effectively take a photo of an equation and get an instant answer, I wonder what the value of number crunchers will be. My philosophy historically is NOT to outlaw capabilities of the calculator, but rather to modify the contest to incorporate advances in technology. Regardless of future capabilities of new calculators, scientists and engineers will always need to apply basic principles of physics and math to understand the world and to model same for the betterment of society. However this is done, the Calculator Applications Contest (or its replacement) will be there to prepare high school students to enter these fields.

DB
Austin TX
2023

Preface to the 2010 Edition

This manual and the companion drill manuals represent a major revision in the UIL Calculator Applications Contest. The process exploded into motion in the middle of 2002 when the hp 32SII calculator went off the market. This caused a flurry of anxiety on the parts of contest participants since for some years this calculator dominated the contest. Hewlett Packard had discontinued popular calculators over the 23 years of the contest, but in the past there was always a new Reverse Polish (RP) version waiting in the wings. This was not the case in 2002. The RP notation lends itself to lightning-fast number punching, particularly valuable on the "number-cruncher" part of the contest. The anxiety focused on what to do when the calculators wore out and what to tell new coaches and students in terms of which calculator to buy.

I spent the fall of 2002 talking with coaches and contestants by email and at all of the fall Superconferences. Some consensus was reached to revise the contest to make it more "calculator-proof" rather than to outlaw specific brands of calculator. It became apparent quickly that it was time to revise the contest *per se*, since the format had not changed in the last 23 years although high school math curricula certainly had. I contacted the Texas Math and Science Coaches Association (TMSCA) in January of 2003 with the intent of identifying a coach to help with the revision process. Brad Friesen from Plains High School accepted the challenge, and the contest is the better for it.

I met with Brad in late January and we agreed on the following plan. We would assemble a Calculator Futures Committee consisting of two types of members. First was the Group Committee whose task was to meet and work out a new contest format. The second group was the Individual Committee whose task was to critique the new format cold without the benefit of any debate or discussion. Diversity was valued in the membership selection. We wanted representation based on varied school size, geographic region, coach experience level, ethnicity and sex. The Group Committee members were: Karol Albus (Whitharral), Dennis Cabaniss (Salado), Oscar Castanada (San Antonio Southwest), Brad Friesen (Plains), Carl Krug (Grady), Gwen Parish (Springlake-Earth) and Fabian Quintana (Pharr-San Juan-Alamo). The Individal Committeee members were: Fraron Holik (Iraan), Cliff McCurdy (Argyle), Faye Parish (Bridge City), Leo Ramirez (McAllen) and Andy Zapata (Azle). The Group Committee met in Austin at the UIL Office on February 17, 2003. Based on the excellent discussion and decisions, I assembled two draft contests in the new format for circulation and critique. Responses were received by March 6, and a presentation was made to the leadership of the TMSCA at their state meet in San Antonio on March 14, 2003. The new format was presented and discussed at the 2003 Fall Superconferences.

This Contest Manual describes this new format. The main features relative to the old format are: reduction in the number of crunchers, an increase in the number of stated/geometry problems, addition of pre-calculus and calculus as subject categories for problems, and a formal study list of stated/geometry problems to replace the old Page 6 "repeat" problems.

I have created 12 appendices to this Contest Manual that condense the required memory work for the contest and provide summary information on various stated and geometry problem types. It is hoped that this localized resource will be an asset to mastery of the contest subject matter. Chapters 2 and 3 were written ten years ago based on a survey of coaches across the state and deal with getting started as a contestant and as a coach. The subject matter was still largely relevant, but some updating was required. I am thankful to Oscar R. Castañeda/Dragon Calculators from San Antonio Southwest High School, Brad Friesen from Plains High School and Faye Parish from Bridge City High School for their critical proofing and revision of these chapters.

Last, I want to acknowledge Dr. John Cogdell, for whom this manual is dedicated. John was the creator of the Calculator Applications Contest as a replacement for the Slide Rule Contest and served as the Contest Director for 18 years. Since then, John has worked behind the scenes with UIL contest problem generation and answer key creation. By my recollection, he invited me to join as an Associate State Director in the fall of 1980. I have through this period enjoyed immensely working with John on this contest, and I appreciate his friendship and wisdom as a senior Christian brother. John will retire from The University of Texas about the time this manual takes effect. He has been a good mentor on several planes and has been a generous leader for Calculator Applications participants through the birth and maturing of the Contest.

D.L.B.
February 20, 2004
Austin, Texas

Preface to the 1993 Edition

We are pleased to offer this Calculator Applications Contest Manual. We hope that it will be of assistance to students and coaches at all stages of advancement. Included in the manual are our philosophy of the Contest, pointers for persons who are new to the Contest, detailed descriptions of the layout, problem types and their solution, and various tutorials on themes relevant to successful participation in the Contest.

Much of the advice we offer here comes from Calculator Applications coaches who participated in a survey this year. We appreciate those who responded to our survey. Our "Thank You" is the incorporation of your comments and suggestions into the body of this Manual. The Texas Math and Science Coaches Association officers and Board of Regents were once again of great value to us and provided a great service through their willingness to proof the survey instrument before its public distribution.

This Contest Manual and the accompanying drill manuals are new and undoubtedly contain various types of numerical, typographical and concept errors. Should you discover an error in one of the manuals, we would appreciate your dropping a note to one of us or the UIL office so we can correct it in future editions.

In Chapter III we discuss incentives to coaches and students who are involved in the Calculator Applications Contest. The reason we are pleased to serve as your Contest Directors is simply that we enjoy it. It is a meaningful way to be involved in community service, where the community here is the State of Texas. We both derive great satisfaction from creating, developing and working the stated and geometry problems on the tests. We enjoy each others company. We share vicariously the thrill of the Contest with the students and coaches who practice, compete, win and lose. We are heartened by our association with sincere, caring coaches across the State who value the young persons in their charge. We are honored to lead this Contest.

D.L.B.
J.R.C.

December 28, 1993
Austin, Texas

Table of Contents

Page

Chapter 1

Introduction

A. Purpose

The Calculator Applications Contest is designed to develop and reward skills associated with operation of handheld calculators. The calculator though is a tool, and success in the contest requires knowledge and ability in its operation. The desire is for the best calculator user to win, not the best calculator.

There are three main areas of excellence required to be a successful Calculator Applications Contest competitor. First, one must be able to perform mathematical operations on the calculator accurately and quickly. This includes the number pad, operations like adding and multiplying, all the trigonometric functions, and power functions like yx and log. It also includes use of simple programming to solve equations and statistics features like $\bar{x}$ and Σ. While the contest is not a speed contest per se, time management is critical.

The second area is knowledge of basic equations of mathematics and unit conversions. Examples are the equation for the area of a circle and 1 ft = 12 in. Appendices to this manual list explicitly unit conversions, specific equations and procedures needed for the contest. Contestants are assumed to know these unit conversions, equations and procedures.

The third area is the ability to reason through problems and to formulate and execute solutions. Many problems on the contest require either reading a description of a problem or analyzing a drawing. An important step is identifying clearly what is being asked and developing a procedure to use information provided in the problem statement to obtain an answer.

Overall, the contest is preparatory for persons interested in pursuing careers in engineering. The skill set is precisely that needed in engineering. While professional engineers have more knowledge in a variety of technical areas, they essentially are applying math skills and problem-solving creativity in the same way Calculator Applications Contest competitors do. To that end, the contest directors for this contest and its predecessor, the Slide Rule Contest, have always been engineers.

There is a subtle but important feature embedded in the Contest philosophy which arises from the engineering perspective. The engineering approach presupposes a rational, reasonable perspective. For example, if a geometric problem for a scalene triangle asks for an angle that is clearly shown on the test as acute, then the test taker may assume that it is acute and not obtuse. Some problem solutions may have multiple mathematically correct answers that include obtuse angles (e.g., use of the Law of Sines). An obtuse angle, while correct in a mathematical sense, would be counted as incorrect on the contest based on this engineering perspective. The depiction of the acute angle provides information that the angle is in fact acute and not obtuse. The point here is that engineers work in the real world, and their conclusions must make sense in the real world. For example, most trajectory problems dealing with vertical distance calculations yield both a positive and negative time of flight. Unless the problem explicitly referred to a past event, the negative time has no meaning, and it would be considered incorrect on the Contest, even though it is mathematically correct.

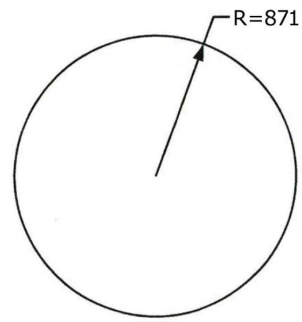
The contestant then, for stated and geometry problems in particular, after obtaining an answer, should ask themselves, “Does this answer make sense relative to the information provided in the problem statement and in reality in the real world in which I live?” When considering these factors, the contestant is entering the world of engineering.

B. Prerequisite Mathematics

Problem solving on the Calculator Applications Contest requires extensive knowledge of algebra, geometry and trigonometry. Every contest has problems dealing with concepts such as matrices and statistics as well as both differential and integral calculus. The types of problems are similar to coverage in standard high school textbooks.

C. Types and Number of Problems

There are three types of problems on the Calculator Applications Contest. First are numerical problems or number crunchers. These problems consist of an equation with numbers and operators, like "(3 + 7)/4". The second type of problem is stated problems or word problems. Here, the form of the problem is words that must be converted or interpreted into an equation that may be solved. An example is, "How many times can 760 be divided by π and still have a positive remainder?" The third problem type is geometry problems, depicted in the figure. They are a drawing of some geometric figure (like a circle) with additional information ("R = 871") and a question about the figure ("Circumference = ?"). Every contest has 35 numerical problems, 21 stated problems, and 14 geometry problems, totaling 70 problems.



Circumference = ?

D. Contestant Skill Set

To be successful, there are general areas of performance. For numerical problems, the ability to punch numbers and operators on the calculator quickly and accurately is critical. Since there can be as many as 10 numbers in a numerical problem, knowledge of how numbers are stored during data entry is important too. Successful contestants treat the numerical problems as a speed contest, punching numbers at a fast pace. The purpose is to maximize remaining time on the more difficult stated/geometry problems. Stated problems require more than the knowledge base to compute answers. They also require the ability to read a problem and to devise a plan to use the given information to generate the answer. The primary skill set for geometry problems is knowledge of the various equations relating lengths to areas and volumes, including trigonometry.

There is a sizable knowledge base that the contestant must master to successfully compete. These are listed as appendices in this manual. The knowledge base is broken into three areas. First is a set of conversion factors (Appendix A) for stated problems. These are values which will not be listed on stated problems even though they may be necessary to obtain a solution. For example, it will be assumed that contestants know that there are 60 minutes in one hour, 12 inches in one foot, etc. The second area is a set of specific equations for stated and geometry problems (Appendices B to L). These are concise summaries of information presented in Chapter 4. Coverage of these topics may or may not be explicitly covered in high school math classes (e.g., scaling problems and logarithmic solutions), and some of the information deals with "conventions", arbitrary assumptions like "the direction of 'up' is positive" and the choice of the basis for a percent change equation. The third area is a procedural summary for working significant digit stated problems (Appendix J). The details of this procedure are given in Chapter 4Miii.

Handheld calculators are becoming increasingly more powerful in terms of the program information that can be stored. Contestants may enter user-created, executable programs prior to the contest and use them during the contest. The storage registers may be populated before the Contest begins, and they will not need to be cleared before the contest begins. All these features may be used by the contestant during the contest. Patches can be downloaded from manufacturer and third-party sites. These downloads and patches are all allowed. Graphing calculators are allowed.

The distinction between handheld calculators and handheld computers is increasingly blurring. The Calculator Applications Contest allows any handheld device that meets the use criteria described in Chapter 2A, including handheld computers. Examples of calculators that have computer features are the ti-Nspire and the hp Prime.

Like almost every endeavor, practice helps. There are two sets of manuals accompanying this Contest Manual to assist in practice. One consists of numerical problems and answers; it provides opportunity to work on speed and accuracy with the number crunchers. The other practice manual lists stated and geometry problems by topical area. These manuals will provide ample opportunity to master the problem types.

E. Scoring the Contest

A problem is attempted if anything (including work product) is written in the answer blank area. There are four ways to consider a problem answer, two for attempted problems and two for non-attempted problems. First, the problem is answered correctly if the correct answer is written in the correct form in the answer blank. The features of a correct answer are detailed in Chapter 2D (Writing Answers). A problem is worked incorrectly if the answer or form of the answer is not correct. A skipped problem is one with nothing written in the answer blank but for which there is at least one attempted problem later in the contest. A remaining problem is one with no answer and with no attempted problem later in the contest.

With the exception of "SD" stated problems, each of the 70 problems is worth 5 points if answered correctly, and contestants are penalized 2 points for incorrect answers. Skipped problems are considered incorrect, but remaining problems receive neither points nor a penalty. As an example, suppose a contestant worked only the first page of a contest, ten problems. They worked Problems 1 through 6 and 10 correctly, worked Problem 7 incorrectly and skipped Problems 8 and 9. Nothing was written in answer blanks for Problems 11 through 70 so these are remaining problems. In this case, the score would be composed of 5 points times 7 problems minus 2 points times three problems, or 35-6=29.

The highest possible score is 5x70 or 350. The lowest possible score is -2x70 or -140. A contestant receives the lowest possible score most easily by scrawling something in the answer blank of Problem 70. This makes Problems 1-69 skipped, and Problem 70 is incorrect.

Most problems require answers to be written with three significant digits. The concept of significant digits (SD) is explained in Chapter 2D. "SD" stated problems are a particular class of stated problem for which both the correct answer AND the correct number of significant digits must be calculated. A problem is denoted as an SD problem by underlined number/s in the problem statement and by "(SD)" written in the answer blank. For SD problems, the answer receives 5 points if written correctly with the correct number of significant digits, and it receives the usual 2 point penalty if written incorrectly or with only one significant digit. If the answer is written correctly but with the wrong number of significant digits, the problem receives 3 points instead of the usual 5 points.

Most experienced contest graders use this formula for calculating grades:

$$\text{Score} = 5\begin{bmatrix}\text{Last}\\ \text{Problem}\\ \text{Attempted}\end{bmatrix} - 7\begin{bmatrix}\text{Number of}\\ \text{Incorrect / Skipped}\\ \text{Problems}\end{bmatrix} - 2\begin{bmatrix}\text{Number of SD Problems}\\ \text{Answered Correctly but}\\ \text{With the Wrong Number}\\ \text{of SDs}\end{bmatrix}$$

For this formula, an SD answer written correctly but with the wrong number of significant digits is not considered to be incorrect relative to the second term in the equation.

Chapter 2

Getting Started

A. Choosing a Calculator

Although there are many calculators on the market which are suitable for use in the Calculator Applications Contest, some care is appropriate in the purchase of your calculator. The calculator should be of the "engineering" type, having the appropriate mathematical functions in addition to add, subtract, multiply, and divide capabilities. Further, the standard scientific functions must be available. These include the trigonometric functions (such as sin, cos, tan) and their inverses, the logarithmic functions (like log, ln, $\sqrt{x}$, x^2, e^x, y^x), and a "pi" key. A root-finder or solver capability is needed for certain types of stated and geometry problems. The calculator should be affordable to you and must be reliable. Some coaches recommend using two different calculators, one with RPN notation (see below) for speed, and a graphing calculator. Graphing calculators are useful for certain types of problem including the transcendental stated problems and some of the calculus problems. For some students, the visual advantages of the graphing calculators will assist in solving speed.

The best calculator for the Calculator Applications Contest is not the most expensive one with lots of features. In fact, the more expensive calculators are a liability because it takes too many keystrokes and therefore too much time to perform necessary calculations. For example, some expensive calculators have an alphabet and can be programmed to perform a series of operations automatically. To change from the "degrees" mode to the "radians" mode (Angular measure is discussed in Chapter 4E in the section on Rates.), these calculators require as many as six keystrokes. Clearly, a simpler calculator with a "deg↔rad" key that toggles between these modes will operate much more efficiently. The same is true when switching from fixed to scientific notation (dealt with in Section D of this chapter). You are referred to the UIL Constitution and Contest Rules for more information on the types of calculators allowed. Basically, any calculator is permitted so long as it does not require auxiliary electric power and it is not custom-modified using noncommercial hardware.

At this writing, there are two different types of calculator systems. These are the RPN type (Reverse Polish Notation) and the algebraic entry type. The RPN calculator operates on numbers like you might do when you do long-hand, column arithmetic. When you add two numbers, for example, you could write the first number, then you write the second number, then you add. If, after writing the numbers, you decide to multiply instead of add, that is possible. Similarly, with an RPN calculator, you enter both numbers, then you perform some mathematical operation. The calculation is actually performed when the operation key is pressed, making it unnecessary to have an "=" key. On RPN calculators, you must have a means to tell the calculator you are finished entering the first number and are ready to enter the second number. This is accomplished by pressing an "Enter" key between the two numbers. Most RPN calculators have an automatic memory feature enabling the calculator to store up to four numbers awaiting a computational command. This four-number memory is called the "stack", and it allows complex numerical calculations to be done with high-speed efficiency since intermediate numbers in a calculation are automatically stored in the stack and automatically retrieved when needed. This eliminates extra keystrokes associated with parentheses and with manually storing and retrieving numbers. Most RPN users find the operating system with its stack to be quite sophisticated and elegant. Details of RPN operation are given in the calculator users manual.

The algebraic entry calculators, as this name suggests, operate on numbers as you would write an algebraic expression. To write the sum of two numbers, for example, you would write the first number, then write plus (+), then write the second number. The sum would be performed when the equals (=) key or another mathematical operation is indicated.

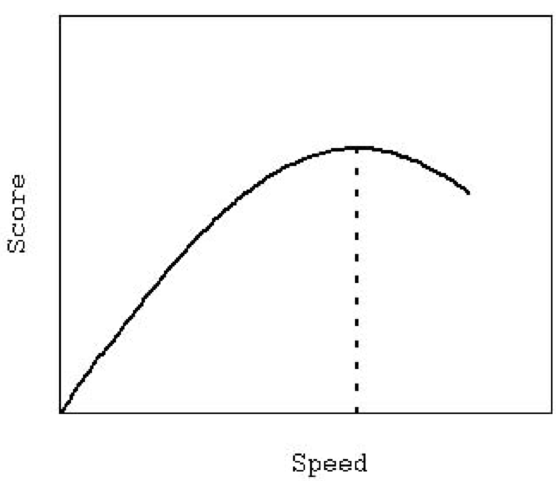
Both types of calculators are widely used and both have their advantages in different applications. The RPN calculators are reputed to solve number crunchers with greater speed, but the solver function is weak compared to the algebraic calculators. Since two calculators are allowed on the contest, some choose to compete with one of each type. The contest has been designed to balance these advantages such that a contestant can successfully compete with either system. It is not the place of the UIL to advise on specific calculators or even specific types of calculators. Hence the student must make up his or her own mind as to what calculator to buy. The UIL Calculator Applications Contest website usually lists in a pie chart the calculators brought to the previous State Meet.

In summary, you should obtain the best calculator you can comfortably afford, for several reasons. For one, you will be using your calculator in non-academic as well as academic activities in addition to the Calculator Applications Contest, and in these uses you will eventually find many applications for the advanced calculator features. Secondly, the calculator stimulates careful thought and mathematical learning, and, the better the calculator, the more good exercise the mind receives. Third, Texas is using certain specific brands of calculator for standardized competency exams, and this may impact which calculator you choose to master. We recommend that you obtain the best calculator you can afford, even if it does not directly benefit you in the Calculator Applications Contest. As discussed above though, too many advanced features may slow down operation which is a real liability on the Calculator Applications Contest, so some compromise may be necessary.

B. Using the Calculator

The place to start is in gaining skill (accuracy and speed) in calculation problems of the type which appear on the Calculator Applications Contest. This is largely a matter of practice, but intelligent practice and not mere mechanical practice is encouraged. Here are some pointers toward intelligent practice.

First, learn how to work your calculator. Read the instruction materials provided by the manufacturer and work the examples in these materials. Skip those sections which deal with mathematical functions and operations which are beyond your present knowledge. If you find the entire booklet too advanced for your present mathematical understanding, get some help from another person in learning how to perform operations of addition, subtraction, multiplication, division, squares and square roots. These are the operations required on the first four pages of the Calculator Applications Contest and these calculations are handled in a simple fashion by all calculators. You should acquire considerable skill in doing problems of this sort before progressing to the more advanced problems.



Second, realize that the Calculator Applications Contest is a speed contest. Thus, soon you will have to modify the "hunt and peck" method and adopt the approach of not looking at the calculator while entering numbers and perhaps operating the common four functions: +, –, x, ÷. We would recommend that you begin this manner of practice soon after you become confident as to which buttons to push.

Third, realize that accuracy is very important and should not be sacrificed for speed. At any point in your development as a Calculator Applications contestant, your score versus speed curve will be as shown in the figure. Here we have sketched speed as the abscissa (horizontal axis) and score as the ordinate (the vertical axis). The sketch shows that low speeds lead to low scores for obvious reasons, but that high speeds can also lead to low scores, since accuracy inevitably suffers. Your best scores will come when you work at speeds in the middle, not too fast and not too slow. Normally during practice sessions, you will want to work faster than the optimum, so as to push yourself to increase your speed. During an actual contest, you should work as near the top of the curve as you can, since the highest possible score is your goal in that situation. Of course, only through practice can you establish your personal score/speed curve, and only through practice can you improve it.

Another aspect of intelligent practice is to use all the convenience features of the calculator. All calculators have memory capabilities which can be used to advantage in calculations involving repeated numbers. Part of the task of learning how your calculator works is to experiment with such features through trying several ways to perform a particular sequence of calculations. If a number is repeated several places in a number cruncher problem, you might find it faster to store the number and recall it rather than entering it by hand every time.

The companion manual, UIL Calculator Applications Practice Manual for Numerical Problems, offers 26 versions of the numerical problems on all seven pages of the contest, or almost 1,000 problems. This is a valuable aid in improving proficiency in accuracy and speed, particularly for the UIL Calculator Applications Contest. This test has repeatable forms of equations which you will recognize as you gain experience in the contest. The numbers are always different. For example, for Problem 1 there are five forms. One form involves multiplying two numbers together and adding a third number to it. Practice Manual Pages DrD, DrI, DrP, and DrU have this form for Problem 1. On average, one out of every five tests has this form for Problem 1, but the numbers being operated on will always be different. The point is that as you work through the Practice Manual, you will be learning the standard forms of the numerical problems you will see on future UIL Calculator Applications Contests. The reason there are limited numbers of forms for each numerical problem is that these problems, along with their answers, are computer generated.

C. First Steps

Once you have chosen a calculator and are familiar with its operation, you are ready to start preparing for the Calculator Applications Contest. So, where do you start? We recommend that you begin by learning how to operate your calculator in a "two-finger" or higher mode, perhaps using both hands. The trick here is to use more than one finger to punch buttons. It may seem uncomfortable at first, but eventually you will be able to work problems faster than "one-finger" operators. Next, learn how to write answers in three-significant-digit notation. This is detailed in the next section. The easiest way to master writing answers is to learn about scientific notation and to set your calculator to the proper setting. Now, locate a UIL test and look over it to become familiar with the location of numerical, stated and geometry problems. It may seem really boring, but you should also read and study the rules of the Contest as contained in a current copy of the UIL Constitution and Contest Rules.

Once you can write answers, get a copy of the UIL Calculator Applications Practice Manual for Numerical Problems and work all the Page 1 problem sheets. Don't worry about time at first; concentrate on accuracy. If this manual is not available, get a bunch of old UIL tests and work only the numerical problems on Page 1. Once you have mastered the Page 1 numerical problems, go on to the Page 2 numerical problems. Repeat your practice for Page 3 and Page 4, concentrating only on the numerical problems.

After this, you may be sick of doing number crunchers! However, if you can do the first four pages of numerical problems perfectly in 30 minutes, your score will be 70. Now, the best way to improve your score is to master the first four geometry problems. The first two problems are simple, usually one-step solutions for simple geometric shapes. The second two problems involve right triangles. They are easy to do, and you would benefit from the UIL Practice Manual for Stated and Geometry Problems, a compact source for lots of these problems to practice. Working these 4 problems correctly in the 30 minute test time raises your score to 98.

At this point, you should study Section E of this chapter on how to maximize your score. You should begin learning about the stated problems on the Contest. At this point, you may start taking practice tests for speed and accuracy. It could be fun and useful to chart your progress by recording your score and plotting it versus time as you progress. Continue to learn and drill the numerical problems on Pages 5 through 7, keeping in mind that to maximize your score, you need to become proficient in working stated problems and the solid geometry problems on Pages 3 and 5. Take every opportunity you have to test yourself against other students in your school and at practice meets. And all along the way, don't get discouraged, especially early in your endeavors!

You might consider yourself an “advanced” contestant when in the allotted 30 minutes for the contest, you can work numerical problems on all 7 pages, the first 3 pages of geometry problems and any 9 of the stated problems on Pages 1-6. If worked correctly, your score would be 210. Realistically, no one works all the problems completely correctly. Each incorrect or skipped problem is now worth 7 points due to the 2 point penalty for incorrect and skipped problems. Scores in the area of 250 are competitive with State Meet contenders. Your goal now is to continue to learn how to work more stated and geometry problems. This contest manual gives information on all problem types. You might consider reading about some type of problem (for example, acceleration problems in Section 4Eii of this manual). Follow this with practice by working associated problems in the drill manual for stated problems. The ultimate goal is to work all 70 problems correctly, but don’t be discouraged if this doesn’t happen immediately; no one, even at the State Meet, has ever made a perfect score on a contest.

D. Writing Answers (Concept of Significant Digits)

Suppose you worked this problem on a Calculator Applications test.

$$6.57 + \pi - 3.55/7$$

The number which appears on your calculator display would probably look something like this: 9.204449797. This is the correct answer in terms of button pushing, but if you wrote this in the answer blank on a UIL Calculator Applications test, it would be counted wrong! The reason is that – with the exception of "special" stated problems discussed later – we demand all answers to be written with three significant digits.

So, what is a significant digit (sometimes noted as "SD")? First, let's define what a digit is. It is simply one of the integers between 0 and 9. There are ten different digits. A number may have any number of digits. The table below lists some numbers and the number of digits in each. The negative or positive sign is not a digit.

All the digits in the numbers in the table are important, but they serve two different purposes. Almost all of the digits are used to define the number, but some of the zeros are used just to keep the decimal point in the right place. These zeros are easily identified because they disappear when the number is written in scientific notation. The third column of the table shows the numbers in scientific notation. The "x10^n" part of the notation is not considered when counting digits. See how in some numbers the zeros disappear? These zeros are not significant digits by definition. All the remaining digits are significant digits. Indeed, all the digits of a number written in scientific notation are significant. The first two numbers in the table each have three significant digits and no nonsignificant zeros. The third number in the table has 5 significant digits and 3 nonsignificant zeros. The nonsignificant zeros in a very small number are often called "leading zeros" because they "lead" the number.

The number 597,400 has two zeros at the end of it. These are called "trailing zeros". Just looking at this number, we don't know whether these zeros are significant or not. Some textbooks have a convention for noting a significant trailing zero to distinguish it from a nonsignificant trailing zero. We do not do this on the Calculator Applications Contest. Rather, we adopt the convention that all trailing zeros to the left of the decimal point are not significant and that all trailing zeros to the right of the decimal point are significant. From this rule, 597,400 has four significant digits (its zeros are to the left of the decimal point), and 6.600, the next number, also has four significant digits (its zeros are to the right of the decimal point). You might ask, "Is it possible to write 597,400 with more than four significant digits?" The answer is "Yes", but doing this on a Calculator Applications Contest requires that the number be written in scientific notation. This moves all trailing zeros to the right of the decimal point by the nature of the notation, so the significance of trailing zeros becomes explicit: 5.97400x10^5 (6SD) or 5.9740x10^5 (5SD).

Significant Digits Analysis

Number in Fixed Notation	Number of Digits	Number in Scientific Notation	Number of Significant Digits	Number Written to 3SD Precision
753	3	7.53×10^{2}	3	753
–6.94	3	-6.94×10^{0}	3	–6.94
–0.0059143	8	-5.9143×10^{-3}	5	–0.00591
597,400	6	5.974×10^{5}	4	597,000
6.600	4	6.600×10^{0}	4	6.60

In scoring the Calculator Applications Contest, there is some ambiguity in writing certain answers in fixed notation. For example, suppose an answer appears on the calculator as 499.975634. Writing this answer with three significant zeros yields 500, but just looking at the number, we don't know whether it is in fact a one, two or three significant digit number. On the contest, when this ambiguity exists, we assume the contestant intended to write the answer with the correct number of significant digits. That is, we give the contestant the benefit of any doubt when it comes to answer writing. It is emphasized though that this issue does not occur when writing answers in scientific notation. We would count this answer correct if written as 5.00×10^{2}, and we would count incorrect the one and two significant digit answers, $5. \times 10^{2}$ and 5.0×10^{2}, respectively.

Writing answers with specific numbers of significant digits introduces the idea of how we write numbers in science and engineering. Why wasn't the number 6.600 simply written as 6.6? It is the same number, isn't it? Or for that matter, why not write it as 6.6000 or 6.6000000? The reason is that all these numbers carry a specific meaning different for each number, at least in science and engineering. The difference has to do not so much with the absolute size of the number but rather with the accuracy or precision with which we measured it. Almost everything in the applied sciences and engineering is measured. Just try to think of something that's not! We use a ruler to measure length, a thermometer to measure temperature, a balance to measure weight and a stopwatch to measure time. The accuracy of the number we obtain depends on the quality of our measuring instrument. Even when we do a calculation, it almost always requires that we use some other measured quantity. For example, we can calculate the circumference of a circle to be exactly πD, but we must measure the diameter D before we can do the calculation.

Here's another example. Suppose we gave you an unsharpened pencil and asked you how long it was. You might look at it and say, "It looks 7 inches long to me." Not bad. You would not say, "It is 6.600142 inches long." At least, if you did respond that way, we would have a good laugh, because we know that you can't measure a length with your eyeball to the accuracy of millionth of an inch. That's only about the distance of a line of 100 atoms! Assuming then that you respond responsibly, using your eyeball as a ruler provides only about one significant digit for the pencil length.

Suppose we then gave you a 12-inch ruler, and you measured the pencil. You might then tell us that it was 6.6 inches long. The ruler is better than your eyeball, and we have a new measurement with two significant digits of accuracy. We might then take away the ruler and give you a set of vernier calipers. These typically measure to the nearest thousandth of an inch. With instruction on how to use a vernier caliper, you measure the pencil to be 6.600 inches long. We now have four significant digits, and the two zeros are important. In this case, they tell us or anyone else who is interested that the pencil length was measured with a device that could discern thousandths of an inch. 6.600 is different from 6.6 in this regard, and the former number carries more information.

For this reason, zeros to the right of the decimal point and at the end of the number say something about the quality or precision with which the number is measured. Now you may be wondering what this has to do with taking a test, since we don't make you measure anything during the test. There are two applications of this on Calculator Applications tests. First, a type of "special" stated problem is called a significant-digit problem. You must work with numbers of varying significant digits and report the answer correctly both in terms of the accurate answer but also with the right number of significant digits! The details appear later in Chapter 4Miii of this manual, but it should be mentioned here that when you write an answer in fixed notation with trailing zeros, we give you the benefit of the doubt respecting whether the trailing zeros are significant. That is, if you had written 597,400 in the answer blank for an SD problem, and the correct answer was $5.9740 \text{x} 10^5$ (5SD), we would count your answer correct, assuming that you meant to write the answer with five significant digits. If you were thinking the answer had only four significant digits and wrote it as 597,400, we count it correct anyway, and you lucked out! The same is true if you thought it should have been written with six significant digits. That's what we mean when we say we give you the benefit of the doubt.

The second place where measurement is important is more indirect. Before 1980, the pre-engineering contest was a slide rule contest. You may never have seen a slide rule and probably are not familiar with its operation. That's okay. Slide rules are like rulers in that the operator has to read off numbers from a calibrated rule. The most accuracy to be expected was three significant digits. So on the slide rule contest, all answers were to be written to three significant digits, the maximum expected accuracy of the instrument. With calculators, this limitation no longer exists, but the three-significant-digit rule for answer writing has been retained. This is not only for nostalgic reasons. Three significant digits is enough to tell the grader that you punched the keys correctly, and it keeps you from wasting time writing down a bunch of digits in every answer blank. Writing answers with three significant digits also demonstrates that you know what significant digits are!

The last column of the table lists the numbers rounded off to three significant digits. If you solved a calculator problem and were ready to write the answer in the answer blank, this is the form it should take. Excepting certain "special" stated problems dealt with in Chapter 4M, all answers are to be written with three significant digits. This requires rounding. We use the normal convention of rounding up when the fourth significant digit is 5 through 9 and rounding down when the fourth significant digit is 0 through 4.

Unacceptable notation generally involves errors in writing the number in scientific notation. The format for three-significant-digit numbers in scientific notation is "$\text{M.NNx}10^e$", where M, N and e are digits, and "M" is not equal to zero. Consider the answer $-1.23 \text{x} 10^4$. The following are some incorrect forms of the answer which contribute –2 points to the test score. The "Correct" and "Correct But Not Recommended" forms net +5 points.

$-1.23 \text{x} 10^4$ Correct answer in answer key

Incorrect Answers		Correct Answers
$-12.3 \text{x} 10^3$	Coefficient is greater than 9.99	$-1.22 \text{x} 10^4$, $-1.23 \text{x} 10^4$, $-1.24 \text{x} 10^4$
$-0.123 \text{x} 10^5$	M = 0, too many Ns	–12,200 or –12,300 or –12,400
$1.23 \text{x} 10^4$	Negative sign is missing	<u>Correct But Not Recommended</u>
$-1.2 \text{x} 10^4$	Only 2 significant digits	$-1.22 \text{x} 10^{04}$, $-1.24 \text{x} 10^{04}$
$-1.230 \text{x} 10^4$	Four significant digits	–12,200. or –12,300. or –12,400.

–1.23E4, $-1.23(10)^4$, $-1.23{*}10^4$, $-1.23 \cdot 10^4$ These notations are not allowed

From this list of acceptable answers, a ±1 error in the third significant digit is allowed, which means that you generally don't have to be overly concerned with rounding. (This is not the case with "special" stated problems!) If the correct answer on the answer key were 12.3, any of the three answers, 12.2, 12.3 and 12.4 would be counted as correct. They would also be counted as correct if they were written in proper scientific notation: $1.22 \text{x} 10^1$,

$1.23x10^1$, $1.24x10^1$. Any other answer besides these six would be counted incorrect, because it would be too far away from the correct number in terms of accuracy, or it was written with the wrong number of significant digits, or it was written with unacceptable notation.

Some other answer notation which we consider "correct but not recommended" and "correct" appear below.

Correct But Not Recommended		Correct
Number	Preferred notation	.174 or 0.174
$1.23x10^0$	1.23 (fixed notation)	
4.56x10	45.6 or $4.56x10^1$	
789.	789 (drop the unnecessary decimal)	

A worksheet on significant digits appears at the end of this chapter. The answers are at the end of the manual in Appendix M. The intent is for you to write the numbers with three significant digits in both fixed and scientific notation.

Answers for almost all problems on the Calculator Applications Contest may be written in either fixed notation or scientific notation. The choice is up to the contestant. However, for two kinds of "special" stated problems, scientific notation is not allowed. These are integer problems, where the answer is written to the "ones" place regardless of the number of significant digits, and dollar-sign problems, in which the answer is written to the nearest penny, again without regard to the number of significant digits. The reason scientific notation is not allowed is that when we encounter integers and currency numbers in our daily lives, they are virtually never written in scientific notation.

Finally, keep in mind that you must effectively use the 30 minutes to perform your best on the Contest. This is true for writing answers. Because scientific notation has extra characters, namely, the "$x10^e$", for many numbers, it is faster to write the answer in fixed notation. Obviously, for an extremely small or large number, it will be faster to write the answer in scientific notation. The point is, you should write answers in a form that is fastest for you.

E. Maximizing Your Score

You may wonder what the best strategy is to get a high target score. The table at the end of this chapter is a guide to help you achieve your target score. The basis of the table is that working numerical problems is quicker than working stated and geometry problems. Also, we know that Geometry Problems 9, 10, 19, and 20 are very easy to work. Knowing solid geometry, Problems 29 and 30 may not be too difficult either. The assumption is that you work all problems correctly, and for simplicity's sake, no SD stated problems are included. These are probably not good assumptions in reality, but the table still gives a suggested easiest approach to obtain a given score.

Suppose you wanted to break 150. From the table, you could either work the first six pages of numerical problems and eight stated or geometry problems, or the first five pages of numerical problems along with ten stated and geometry problems. Which you choose depends on how good you are at working stated and geometry problems.

The table also helps some with strategy. You may be tempted to work all the numerical (or "number cruncher") problems and then work geometry and stated problems in the time remaining. This may not be the best way to obtain the highest possible score. For example, working all seven pages of numerical problems but no stated or geometry problems gives a score of 5(35) - 2(35) or 105. From the table, you can beat this score by working all problems on only the first three pages to get a score of 150!

Many students work the number crunchers first, then the easiest geometry problems and then the easiest stated problems. This is okay, but you might consider attempting the Page 1 stated problems where they fall on the test. The same goes for the first four geometry problems. This strategy gives your hands a break without requiring too much thinking. Some contestants do this in effect by working 20 minutes on the numerical problems and then spending the remaining time on the stated and geometry problems. Similarly, some students spend the first ten minutes on the stated and geometry problems when they are at their freshest before going on to the numerical problems.

If you want to do better than 245, you should do the problems to get a score of 245 and then continue working stated and geometry problems. Once you have worked the problems to get a score of 245, each additional skipped problem worked correctly will add 7 points to your score.

Solving the stated and geometry problems involves three steps: reading and understanding the problem; recognizing a solution approach; and actually solving the problem to get an answer. For many more advanced contestants, the first and last steps are mastered before the second step. For help in recognizing a solution approach, in addition to lots of practice, you might consider working with others, discussing problems and their solution.

Maximizing Your Score

To Get a Score of	Work All Num. on Pages	Work These Geom. Problems	Work This Many Stated's
54	1-2	2	0
83	1-3	4	0
98	1-4	4	0
104	1-3	4	3
113	1-5	4	0
119	1-4	4	3
128	1-6	4	0
133	All	4	0
134	1-5	4	3
149	1-6	4	3
154	All	4	3
175	All	4	6
196	All	4	9
210	All	6	9
231	All	6	12
245	All	8	12
294	All	12	15
308	All	All	15

It is reasonable but incorrect to assume that the problems on the Calculator Applications Contest move from the simplest to the hardest. For most contestants, the easiest problems are the number crunchers. The Page 2 number crunchers actually require the more keystrokes than the other pages, followed by Page 3. For the stated and geometry problems, one might rank them from <u>easiest to hardest</u> in the following order, assuming that the contestant has the knowledge and background to work the problem. Included in this consideration is the amount of high-level reasoning required to formulate a solution strategy. More details on specific problem types is found in Chapters 4 (Stated Problems) and 5 (Geometry Problems).

Page 1 Stated Problems
Page 1 Geometry Problems
Page 2 Geometry Problems
Page 2 Stated Problems
Page 3 Geometry Problems
Problem 58 (Matrix)
Problem 62 (Logarithmic Solution)
Problem 48 (Solver)
Problem 47 (Linear Regression)
Problem 39 (Triangles and Circles)
Page 3 Stated Problems
Problem 56 (Basic Calculus)
Problem 46 (Scaling)
Problem 40 (Laws of Sines/Cosines)
Page 5 Geometry Problems
Problem 59 (Calculus Geometry)
Problem 60 (Difficult Plane Geometry)
Page 6 Geometry Problems
Page 4 Stated Problems
Problem 61 (Difficult Stated Problem)
Problem 63 (Difficult Stated Problem)
Problem 57 (Calculus Applications)

PRACTICE SHEET FOR WRITING ANSWERS WITH THREE SIGNIFICANT DIGITS

(Write each number with three significant digits in fixed and scientific notation - Answers are in Appendix M)

Number	Fixed Notation	Scientific Notation
64.2528547		
–0.004851202		
–28.33408672		
–0.058566597		
–0.091154255		
0.351614191		
–0.000552416		
0.00099347		
0.00090227		
0.003125191		
–0.0000576415		
–0.070929216		
0.000942375		
–2542.145292		
0.526521273		
0.458663032		
–0.000733427		
–0.055170573		
–0.000852177		
0.842572564		
–13.94580006		
–0.04111643		
28.60622804		
–408.4813209		
–0.501230757		
0.000007568		

Number	Fixed Notation	Scientific Notation
–2527.810821		
0.000056401		
–0.511232545		
–5.558254255		
–0.000038651		
–0.008498004		
–0.000208107		
–872.8743858		
5.243578055		
–0.620209752		
0.000970094		
–0.000056258		
–8181.138157		
–0.589933534		
–0.004832374		
0.006656252		
–3.354455156		
–0.000036167		
–0.006600091		
–0.991647014		
–0.000037683		
0.924285204		
–0.578585854		
–0.267678559		
0.000348063		
–0.098086458		

Chapter 3

To The Coach

A. Introduction

This section is for the high school teacher or administrator who is new to the Calculator Applications Contest. This Chapter deals with getting started and organizing students who are prepared to compete in the contest. It relied on coaches from across the State who responded to an invitation to comment. This was formalized in a survey instrument which was sent to all the school districts in the UIL in 1993. Over 200 responses were returned, and these were tabulated into a database from which information was obtained as needed to complete this section. The 1992-93 Texas Math and Science Coaches Association (TMSCA) officials kindly proofed the survey instrument before its general circulation and made numerous helpful comments. The chapter was reviewed in detail and updated by representatives of the TMSCA in 2023.

B. Personal Incentives

There is a need for coaching. Even more than in athletics, certain schools consistently produce regional and state winners in the UIL Academic Contests – and the reason is not difficult to discover. These schools have gifted, experienced, and energetic coaches who like to take that trip to Austin every spring. Without these coaches and their contributions, occasional winners would come forth from time to time, but there would be no consistent program.

Developing a calculator team is a lot of work both for the coach and for the students. Fortunately, there are outstanding incentives, both internal and external, which might motivate you and your students to your best effort. Most of us were attracted to a teaching career by a certain idealism, some mixture of awe at the learning abilities of students and confidence that we had something personal – in addition to instructional content – to offer. Admittedly, that idealism becomes dimmed through the massive amount of routine work required in instructional tasks and the necessary "averageness" of most students. The UIL Academic Contests offer you an opportunity to recover and fulfill some of that idealism which first drew you into the field of teaching, for these contests offer you focused activities for individual instruction in intense intellectual activity.

The vast majority of Calculator Applications coaches really enjoy coaching. Most have coached for more than 4 years. One of the incentives to coach is financial, although most are quick to point out that the work doesn't pay very well on a per-hour basis. There is a general consensus among coaches that their administration does indeed appreciate and recognize their efforts in UIL coaching. Further, there are financial resources available from the administration to support your coaching efforts.

What makes coaching enjoyable for you, the coach? By far, the greatest motivation is the opportunity to spend time working with top quality students who are themselves motivated and excited about mathematics and problem solving. Many coaches feel that the Calculator Applications Contest provides an alternative to a "gifted and talented" class. In some instances, the level of mathematics skills needed for the Calculator Applications Contest is higher than the coach currently is teaching, so coaching provides a mechanism for teaching at a more interesting and complex level. There is a satisfaction in teaching students mathematical subjects in advance of their presentation in the course curriculum, in part because students catch the enthusiasm of solving problems for the personal satisfaction rather than for a certain grade. Indeed, this parallels the approach of Project-Centered Learning. Working closely with these students gives you an opportunity to get to know them personally outside the classroom setting. You may find that your advice in coaching will really make a difference in terms of a student's learning and advancement. The contest itself provides a simple gage for student progress in terms of the absolute and relative score. Beyond this, coaches report that a real value of getting to know students on a personal level is becoming a role model for them. As such, you have the great pleasure and responsibility of influencing the lives of young people in a positive and lasting way.

Another motivation for coaching is watching students advance and succeed. This means not only helping them learn how to use a calculator effectively in problem solving. It means being a part of the transition in the way a student views himself or herself. The "I can't" becomes the "I can!" Such confidence building is a rewarding part of teaching, and it is built into the Calculator Applications Contest through improved scores and winning. One coach likes to write practice tests that personalize the stated problems to actual students and situations in his school. Done properly, this approach "humanizes" the stated problems and helps transform them from problem solving to situation analyzing. And the students enjoy the increased thrill of recognizing themselves in the test.

Successful coaches report that there is value in the networking among other coaches at meets. This is true not only for issues associated with the UIL contests but also for many matters of interest to high school math and science teachers. The contest and the grading afterwards provide a mechanism for coaches from different schools and districts to interact. Idea sharing, commiserating and relationship building can take place. It is relatively easy to maintain the communication relationship with other coaches after the meets are long over.

There are some other intangible benefits to coaching. Many coaches enjoy the thrill and excitement of competition. Accompanying the student through winning as well as losing teaches valuable lessons and is itself a rewarding experience. A significant number of coaches have had very meaningful experiences: the student's first winning or placing at district or region, the emotional flood in the moments before the winners are announced at a meet, the gleam in the eyes of students who had just bested their "personal best" score on the contest, the consolation and empathy drawn when a contestant had a bad day, learning to be a good sport whether you win or lose. Hours spent traveling to meets and tournaments may sound boring to the novice coach, but when spent with the students, this becomes a real enjoyment. Local meets provide a natural meeting ground and network with other coaches who share your common interests. Along with your students, you become aware of what calculators are available and what new features are coming out. There's a continuing education aspect of the Contest, too. Several coaches relate that coaching keeps them on their toes in problem solving and acts as an academic hone, maintaining a sharp edge on their own problem-solving skills.

C. Student Incentives and Motivation

As a coach you can offer incentives to your students to motivate their best work in UIL activities. Many incentives used by coaches do not involve spending money, and they can be of great value to the students. The personalized attention students receive from a teacher who really cares about them is a strong motivation for participation and should not be undervalued. Like many coaches, students enjoy winning. Friendly competition among your own students works, and improving scores with practice is also a great personal motivator. There are varying levels of peer respect that are great incentives for excelling. The better students in your school will be proud of that achievement, and this serves as an ego booster. Clearly, as a coach, you must be careful when bringing incentives to the students that building up/extolling one student is not accomplished at the expense of or by tearing down another student. This defeats the goals of the UIL, it is not good sportsmanship, and, worst of all, it disparages the intrinsic worth of individuals you should be serving through the educational process.

Students often self reward as they feel the personal satisfaction of success in problem solving. It becomes an ego boost to learn advanced mathematical concepts ahead of their presentation in the curriculum. Done properly, the team approach allows leadership and mentoring opportunities for upperclassmen as well as a feeling of privilege for lower classmen.

As the coach, you can set achievable goals for individuals and then reward advancement appropriately. Personal praise is a time-tested form of recognition that students value. When students do well at a meet or at official UIL meets, broader recognition is warranted. "No cost" ways to do this are school-wide audio or visual announcements, running articles in the school and/or local newspaper, recognition at the PTO meeting or at a school board meeting. A personal note of congratulations from the school principal is also very meaningful to most students. Don't forget to have your students' picture included in the school yearbook, a job made easier by organizing into a team (see next section). Relatively low-cost recognition mechanisms include locker posters, banners in the hallway, or even yard signs for winners. A number of coaches have a bulletin board or display case at their school which can be used for a variety of purposes including recognition. Posting pictures of winners as well as the entire calculator team will bring attention to the participants. Candid photographs from activities like invitational meets, fund raisers, and local practice sessions will help publicize the event. Making a scrapbook to archive these activities is useful since the scrapbook can be used for recruiting and bonding students as well as documenting your considerable efforts. Posting of meet results, award certificates, medals, trophies, etc. in a display case is also a possibility. Many schools already have an academic "honors board" or its equivalent, and you might investigate using it for these purposes, too.

There are other incentives for students, but these involve more financial resources. You might give an "end-of-the-year" award to your contestants, like a certificate, trophy or medal. The most popular form of recognition centers around academic letters and letter jackets. The form is varied across the State. Besides the traditional letter and jacket, coaches report giving patches, sweaters and even blankets. The qualifications for these items also varies. Many students receive these awards for advancing to Region. Another common group is seniors who previously competed at UIL District on one or more occasions, or anyone who competes several times at UIL District.

In addition to these awards, you might consider another reward incentive. Some schools give calculators or a "free meal" to top contestants. In one school, the faculty host a breakfast for all academic contestants just before UIL District. This is a great "send off" for the students. Many schools have an annual academic awards assembly or banquet. This is a good opportunity to recognize your students in the larger context of academics. Depending on your specific situation, you may need to arrange offering of academic "points" as part of your participation in the school awards program. If you develop a math team, you might think about having logo T-shirts made. Students could buy them to mitigate the cost barrier, but you could also give them to students who succeed by winning or improving to a certain level of proficiency in the Contest.

Respecting giving awards to students, there are limitations imposed by the UIL. This is understandable since we do not want to turn our students into professional competitors. The current rules concerning what students may receive are listed in the UIL Constitution and Rules under a section on Awards. Of tangential interest, the same section addresses limitations on awards that a coach or sponsor may derive from sponsoring the Contest.

Students love the adventure of traveling to meets, and this is another incentive for participation. While on a trip and time permitting, you might try a side trip to an amusement park, shopping mall or other attraction in the area of the meet. The kids will certainly enjoy it, and it will help heighten the degree of excitement associated with traveling. One of the largest and most popular invitational meets in the State is the TMSCA State Meet. At this writing, it is held in San Antonio the last Saturday before UIL Academic District Week.

The Texas Interscholastic League Foundation offers scholarships to qualified students, and students must advance to the State Meet to be eligible to apply. Over one million dollars in scholarships are awarded annually to about 500 students. It is not necessary to win at State, only to participate. The TMSCA currently gives 30 scholarships at its state meet, six awards for each division. Scholarships are also available to National Merit Scholars and other students who score well on certain standardized tests – and it is well established that the UIL Contests develop the mental skills which benefit students in the standardized college-admission tests like the SAT and ACT.

D. Recruiting Students

Most experienced coaches are sponsors for more than one UIL Contest. Our experience is that the most organized programs have a math and science team including students who compete in UIL Number Sense, Science, Calculator Applications, Computer Science, Mathematics, and possibly Robotics. For most coaches, recruiting students to compete in the Calculator Applications Contest is part of a larger recruitment plan. For this reason, our suggestions on how you might go about recruiting and motivating students for Calculator Applications will probably also work in general for all these contests.

It is best to recruit potential contestants as freshmen, or even before they enter high school, and work with them up to four years - a wonderful opportunity to develop a personal relationship with the students. This is best accomplished by working with teachers in middle schools or junior high schools feeding your high school. There is a UIL Calculator Applications Contest for the middle school level, and perhaps you could develop a network of coaches in the feeder schools. In larger schools, about one quarter of the coaches develop a contact at one or more of the feeder schools who can recommend who in the incoming class might be inclined to excel in Calculator Applications. This is less popular in the smaller schools across the State. Remember when recruiting that the Contest requires knowledge of trigonometry, plane/solid geometry, matrix algebra, and calculus. If you have a young team, they may need to be patient until these subjects are covered in a classroom setting, or you could develop unit teaching packets to be used at practice based on this contest manual. Chapter 4 of this manual provides tutorials on the stated problems, and Chapter 5 covers geometry problems. The appendices might be a good starting point to assess student knowledge since these represent concise knowledge areas needed for the contest.

The most popular method coaches use to recruit students is making announcements in classes, either their own or a colleague's. Your identifying who the best students are and hunting them down personally is also an effective recruitment tool. Talk to other teachers about who they think would benefit from participation in this problem-solving activity. The student with the best math grades is not necessarily the best student to recruit. Some coaches look for students who excel in typewriting, keyboarding or piano as well as math. Others suggest looking for musicians since they supposedly are good at memorizing. (This is countered by some coaches who say that band members have so many demands on their time that they have no time to be involved in other academic contests.) It seems that the best student to recruit is one who is not only competent academically to excel but who is also willing to commit time to develop skills necessary for success.

Once you have a calculator team, the students themselves can help to convince other students to join up. Many coaches use school-wide announcements of openings in their Calculator Applications/Math team to publicize the activity. The previous section on "Incentives to the Students" may be helpful for students when they ask about the benefits of participation.

E. Setting a Practice Routine

Many coaches act as a resource person for students participating in the Calculator Applications Contest. They provide advice, guidance and materials to students who come by during the school day. Coaches also meet with students as a group, perhaps a math and science team where UIL Science, Number Sense, Math and/or Computer Science are practiced in addition to Calculator Applications. The most popular meeting time varies with school size. Small schools have practice sessions as part of a regular class during the day. Examples are study hall, home room, advisory period, the tutorial period, and the activity period. Increasingly, schools are writing UIL academic activities into their Gifted and Talented programs in the form of a special period to practice for these events. Larger schools tend to meet right before or right after school for a short practice session. Another popular time is during lunch.

Practice sessions organized by the coach need not be lengthy. Most coaches, even the most successful ones, do not meet more than a few hours per week, even in the intense preparatory weeks preceding UIL District. Flexibility in the practice time maximizes participation. One successful coach has a set practice meeting followed by a “late-bird” practice period for students involved in other extracurricular activities like band and sports. Initially, you should talk with students about the reasons they would benefit from participation (see above). You might pass out a test so students could see what the academic coverage is. Significant areas where coaching is really helpful are how to write answers, the strategy of how to achieve a target score, the rules of the UIL Calculator Applications Contest, how to identify and work the various types of stated and geometry problems, what needs to memorized, the scope of the Contest, how to use the calculator, getting started. All these areas are covered in this manual, the UIL Practice Manuals and the UIL Constitution and Rules.

F. Available Resources

The most important resources are people who can help you develop and advance. The State Contest Director for the UIL Calculator Applications Contest is available to answer questions about problems and other aspects of the contest, but you should be aware that the Director has little direct knowledge about “in the trenches” issues in a local school setting or about the inner workings of the UIL itself. For the former issues, other successful coaches are an excellent resource. For the latter, you are directed to the UIL Director of Academics. Another people-resource in schools with more mature calculator activities is alumni. They can visit at strategic times through the year to help with recruiting and team building. Alumni have the strongest sense of what it is like to be a student participant and they can offer unique perspectives on the value of the experience.

Throughout this manual we have made repeated reference to a variety of materials that are valuable resources for coaches and students who plan to participate in the UIL Calculator Applications Contest. This Contest Manual, the two companion Practice Manuals and the UIL Constitution and Rules are the primary resources available from the UIL dealing with the nature of the contest and its organization. Calculator tests from the previous academic year are also available from the UIL. One of the UIL Practice Manuals deals with numerical problems and contains 26 versions of each page of the test. The other Practice Manual contains almost 1000 stated and geometry problems classified by problem type. The Constitution and Contest Rules give specific rules concerning conducting and grading the Calculator Applications Contest. Other topics include the types of calculators allowed, team participation and breaking ties.

The TMSCA (Texas Math and Science Coaches Association, https://www.tmsca.org/) offers a variety of helpful materials through its Resource File. Currently, invitational test materials are available for use in practice meets throughout the academic year. Several successful coaches offer materials through the Resource File: new tests, starter kits for new coaches, step-by-step solutions to UIL stated and geometry problems, practice sets by topic, etc. Another outstanding resource membership in the TMSCA affords is the monthly electronic newsletter. The newsletter contains information on upcoming practice meets around the State, results of recent practice meets, news of interest to math and science teachers at the middle school and high school level, and "how to" articles for new and experienced coaches.

For new coaches and students who have never competed in a UIL contest but who are interested in doing so, the Fall UIL Student Activity Conferences become a valuable resource. Conducted on Saturdays in late September to mid-November, these events are typically hosted by regional sites around the State. All UIL academic events are represented in this all-day affair, and students and coaches are afforded an opportunity to look around and learn something about some of the UIL academic events. The various workshops are typically conducted by the UIL State Directors, and this is the custom in Calculator Applications. Historically, for Calculator Applications, there are three sessions, each lasting about 75 minutes. One session is devoted to introducing the Contest to people who know nothing about it. Another session is generally a tutorial on some topic and/or an open discussion of the Contest. The last session is the conducting and grading of a recent UIL test with discussion of selected problems.

G. Tournaments and Meets

Everyone agrees that practice is the key to success in the Calculator Applications Contest. An essential part of the Contest requiring practice is the dynamics of a meet or tournament: the strange place, new faces, unknown contestants, unfamiliar director, performance after traveling... According to the most experienced coaches who have advanced students to State, tournaments and meets are absolutely essential, and you might consider planning early in the year an overnight trip to a meet in another part of the State. The UIL does not sponsor practice meets or tournaments, although historically they have provided tests for individuals who wish to organize a practice meet. High school coaches across the State do organize practice meets, and you are always invited to attend. One of the motivations to the meet organizers is to generate revenue through meet registrations, so clearly, they want lots of people to attend! The best way to learn about invitational meets in your area is to talk to other UIL academics coaches at your school or in your area. Invitational meets always offer more events than just Calculator Applications. At the very least, the other math and science events are offered. The TMSCA offers information and publicity on a state-wide basis (https://www.tmsca.org/). This organization also hosts the largest meet in the State annually just before the first UIL District Meet.

The distance or proximity of the practice meet to your home town is the most important factor in choosing a meet. Most coaches will travel between 100 and 200 miles to attend a practice meet, although some schools in West Texas and the Panhandle travel further. Traveling much farther than this would usually necessitate an overnight stay which becomes costly.

There are other considerations which are important when selecting where to participate in a practice meet. The cost of registration is a factor, and the type of test being offered is also important. The test being given may be an offical UIL invitational meet test, an old UIL test selected from a test pool, a TMSCA test or a "homemade" test. UIL invitational meet tests are written by the UIL Contest Director, they are uncirculated (i.e., new), and they carry the "feel" of the test your students will see at District. These invitational tests are Test A and Test B. An old UIL test also serves this purpose, but you run the risk of your student having seen it before. The TMSCA currently offers invitational meet tests written by the UIL Contest Director: Tests C, D and G. You may also obtain TMSCA tests written by experienced coaches who are actively involved in this organization. In the past, the UIL Contest Director has provided the numerical problems for these tests, and the stated and geometry problems are generated by the TMSCA test writers. The quality of these tests is also very high, and they are available for almost any weekend that a UIL invitational test is not available. It is a special challenge for coaches to take the time to write their own test, a "homemade" test. It is not an easy task, but there is a reward of gaining new perspectives on the Contest.

Going to a meet costs money. Generally, food is an individual expense. Beyond that, most coaches rely on administrative resources to fund travel and registration costs, especially when the meet is part of the UIL competition (District/Region/State). However, there are other means for raising money. The most popular, large-school fund raiser is sponsoring an invitational meet. Otherwise, for the most part, any other expense is absorbed by the students individually. A few coaches will hold special fund raisers. These span from the traditional candy/bake sales and car washes to handling concessions for ball games, selling seats for parades and a special meal like a Sunday lunch somewhere. Very few coaches use a "dues" system or "joining fee" to underwrite expenses.

Once the day of the meet arrives, what do you do to help your students perform as well as possible? You might tell them to get a good night's sleep the night before, but for many students, especially those competing at Region or State, this is understandably impossible. At the meet, it's really too late to prepare for the test. Your goal here is to help the student relax as much as possible and not get overly nervous about the test. There are a number of things you can do, and which of them work for you will depend on the demeanor of the student and your own personality. For overnight travel, experienced coaches recommend working hard up till the day or two before the trip. Staying up all night practicing is generally a set-up for failure, so do something to take you minds off the test. Some recommendations are sight seeing, shopping, taking in a movie or hanging out at a mall. Either before or after the test day, many coaches try to work in a major local attraction like an amusement park.

Well before the test (on the bus, during long waiting periods between contests, etc.) many students engage in time-consuming diversions: the usual iphone diversions, playing cards, reading a book, playing chess, playing a hand-held video game, etc. When you arrive at the test site, find the building and room as well as the nearest restrooms and snack area. Time permitting, your students might take a walk around the test site or do a "large-muscle" activity like touch football, frisbee, etc. Some students may want to stay in the test room for some time to get used to the surroundings; some may get nervous if they spend much time in the test room. At this time, you might pair novice students with experienced contestants, especially if you need to be involved with several contests.

About 30 minutes before the test, you should make sure that the students' calculators are operational. Make sure that pencils are sharpened and pens are available. Some students limber up by working a practice sheet or old test. Most concentrate on numerical problems rather than the stated and geometry problems. Many coaches remind students to write answers correctly, give some reminder of Contest rules, remind the students of the strategy you both agreed on previously, or just are available to answer last-minute questions. You should encourage your students to do as well as they can, to relax and not to be nervous. You might also remind them that you will not be mad if they do not do well, tell them that you are proud of them for their efforts to this point no matter what the outcome is, and remind them that they are supposed to be having fun! In the few minutes before the test begins, you have several options. Some students prefer a few minutes of quiet. Some want total diversion from thinking about the test, so joke telling and kidding around are called for. Remember not to disturb other contestants though! Some coaches may make their students nervous, so the students may be better off if the coach is not in the room just before the test begins.

After the test, most students want to talk about how they felt they performed and to discuss some of the stated and/or geometry problems. This is useful to help students cope with the testing process, but it is easier to be more specific in a debriefing later, after the tests are returned. In the time between the test and the awards ceremony or announcement of winners, students who do not compete in other contests usually get something to eat, take a walk, look around the area, or engage in the time-consuming pastimes described earlier.

Once the winners are announced, you should give hearty congratulations and celebrate your student's winning. If this is not the case, depending on your students' expectations of winning, you may need to talk with them as the situation warrants, either in consolatory or motivating fashion. You should divorce the student's performance on the test from their intrinsic worth as individuals and make it clear that you value their participation regardless of how they did on the test.

Some type of debriefing after the meet is helpful. You might consider having students retake the meet test at the next scheduled practice time. This gives students additional motivation to learn how to work the unworkable problems in hopes of getting a perfect score at the retake.

H. Other Advice

The following advice is for the novice coach. The most popular starting point is to contact an experienced, successful coach in your area and ask a lot of questions. You may feel sheepish about contacting "the competition", but our impression is that coaches by and large are honored to be asked to help, and they perceive the situation not so much as weakening their position as an opportunity to help someone in need and support the Contest and the students it seeks to serve. A north-east Texas coach proudly filed this report. "Ten years ago, a neighboring high school began competing in UIL events. They sent their coaches to meet with ours (in all UIL events). We helped them get started. Now they are our biggest competition!" You can "hook up" with experienced coaches from all over the State by joining the TMSCA. It is not expensive to join, and the benefits to you as a new coach are immeasurable. Appearing as a threat is eliminated if the local, experienced coach is available who competes in a difference UIL Conference based on school size.

Another hidden resource is the experienced students whom you may have inherited. If you ask them for help in learning the Contest, you will achieve several valuable goals. Of course, you will learn about the Contest. But you will also cement the students in the Contest by confirming that they are important to you. You increase their self esteem by saying directly and indirectly that they are important. New recruits won't feel so uncomfortable if you are seen to be in their shoes! They may even work harder if it looks like they have a chance to outscore you in a practice test! This approach to coaching may be personally humbling, but if done right, the benefits to the students and your relationship to them is incredible.

Another crucial way to get up to speed in the Calculator Applications Contest is to attend as many practice meets as you can afford (see above). Usually, coaches are asked and encouraged to grade the tests, so don't be bashful about volunteering. Be sure to ask lots of questions and don't be afraid of appearing ignorant. The key here is to get lots of "insider" experience coupled to accurate grading of the students' tests! Some practice meets treat novice students as a separate category. This is really good for your new students since they can win even though they don't have a lot of experience. The Fall Student Activities Conferences were discussed in the earlier section on Tournaments and Meets. You should attend at least one of these and use the occasion to help interest students in the Contest. Experienced coaches also recommend that you get practice materials from as many sources as possible, set up a time to meet with students and practice, practice, practice!

Obtain a good calculator (talk to other experienced coaches about what is good presently), and learn how to use it yourself. Some experienced students will work with different brands or types of calculator. Initially, you will probably be better off working with just one type. You should take some tests yourself. A positive outlook is also important. Your enthusiasm will be contagious. Don't look for too much improvement too fast, but set obtainable goals and hold your students to them! Concentrate on the proper motivations and incentives for being involved in this Contest. Remember that the Contest serves numerous purposes, two of which are to enhance students' academic skills and to have fun!

When working with a beginning student, refer to the suggestions given in Chapter 2 of this manual on how to get a high score as quickly as possible. Practice is important at all stages of preparation. Many coaches believe that teaming the novice with an experienced peer for practice and training is of great value. Practice meet participation is also a good way to get new students excited about the Contest. It's also very important to consistently encourage the new students at every turn and as much as possible. Some coaches start their students out on the middle school test (district tests are available from the UIL office), although you would be better served in the long run by using the high school practice materials from the outset. If you're a new coach, you might consider following through the same procedure as a novice student, perhaps skipping working all the problems in the Drill Manuals! This way, you will develop a good feel for the Contest and can discover how you can be of greatest help to the students you are/will be coaching. Last, study the section at the end of Chapter 2 on how to maximize a score and use this as a starting position for strategy building in your training sessions.

Suppose you have a student who is bogged down on the numerical problems and can't seem to improve his or her speed. The easy answer encompassing all the recommendations to follow is to practice more. That said, the most popular remedy is to drill one page at a time using old tests and the UIL Calculator Applications Practice Manual for Numerical Problems. Increasing the number of fingers used often helps, and students should be familiar enough with the keyboard that they can accurately punch keys without looking at the calculator. Taking short (less than 12 minutes), timed tests on the numerical problems also helps increase speed. Sometimes the student will use redundant keys in excess. Examples are over-use of parentheses in algebraic calculators and storing numbers in memory when they could roll into the stack of RPN calculators. Page 7 numerical problems have short cuts that reduce dramatically the number of keystrokes involved. We prefer to let you and your students find them on your own, but they typically involve trigonometric identities and series expansions. For some contestants, moving to a different calculator will help with speed eventually. Last, students typically can write answers in fixed notation faster than scientific notation, although it takes more thought. For stated and geometry problems, encourage contestants to understand the problem and formulate a solution strategy *before* starting to solve the problem.

Chapter 4

Stated Problems

A. Introduction

Mathematics text books contain many methods for solving routine mathematical problems such as quadratic equations. Often these are presented as techniques to be mastered or procedures to be followed. For these to be useful, one must have the problem already in mathematical form. The stated problems of the Calculator Applications Contest must be translated into a math problem before one can proceed with mathematical techniques and computations. In these "word problems" a major part of the thinking is directed toward formulating the situation described into a mathematical problem. Most mathematics books contain inadequate instruction about how to turn a "word problem" into a well-defined mathematics problem, which we suspect is one reason why most people dislike "word problems".

The purpose of this section is to demonstrate methods for solving these problems as they currently appear on tests. Solutions to a number of representative problems are worked out in considerable detail. The compendium volume, UIL Calculator Applications Practice Manual for Stated and Geometry Problems, contains hundreds of stated problems grouped in similar fashion as they are presented here, along with the answers. This is a valuable resource for student use in practicing these principles. If you are new to the Contest, you might read through the proceeding sections, one section at a time, and then try working some of the stated problems of the same type in the Practice Manual. Be advised that on a test, the stated problems are generally easiest at the beginning and become increasingly more difficult, the most challenging problems being at the end of the test. (There is a list of problem numbers ordered by difficulty level in Chapter 2.)

An assumption in developing this part of the manual is that learning to work stated problems rapidly requires one to first become skilled at working them without a time limit. Thus, there is no emphasis in this section on pushing for speed, but rather the emphasis is on a thorough understanding of the problem-solving principles employed. Once these principles are mastered, the student is equipped to work other practice problems in the Practice Manual. Speed will come only through practice and a built-up repertoire of problems which the student understands thoroughly enough to recognize the solution quickly.

B. Order of Stated Problems on the Test

It is important to ensure that each Calculator Applications test has a good, representative coverage of the various types of stated problems. Over the years, this has evolved into a tradition of placing certain problem types in specific locations on each UIL test. While the main flow of the test is to move from easier problems to more difficult ones, as seen in Chapter 2E, this is not universally true. For example, Problem 62 (Logarithmic Solutions) on the last page is one of the easier problems on the test. It may be helpful while preparing for the contest to know the placement of stated problems. Here is the list. Each problem type is covered in this chapter. Each test also contains at least one percent problem, one significant-digit problem, one integer problem, and one dollar-sign problem. These are scattered throughout the test.

Page 1	Easy
Page 2	Still Easy
Page 3	Medium Hard
Page 4	Hard

Problem 46	Scaling
Problem 47	Linear Regression
Problem 48	Solver
Problem 56	Calculus Basics
Problem 57	Calculus Application
Problem 58	Matrix Algebra
Problem 61	Hard
Problem 62	Logarithmic Solution (Large and Small Answers)
Problem 63	Trajectory

C. Problems Involving Mainly Translation

It may sound strange to talk about translating a stated problem into a mathematical form. We think of translating from German to English, or from Arabic into French, but from English into Math? The point is, of course, that it does no good to type words into your calculator. The calculator can only work with numbers (like 0.05 and –76.9), operations (like +, –, x, ÷), and functions (like sine, log, square root). The problem-solving skill of stated problems is to demonstrate a cognitive ability to turn the words on the page into an equation or equations to be solved. Stated problems involving mainly translation are the easiest to solve, so we begin here. An example of a translation problem is, "What is sixteen times the sum of pi and –4.5?" We solve this problem by a more or less rigid translation. "What" implies an unknown, and we may replace it with a variable such as A. "Is" is always translated by an equals sign, "=". "Sixteen" becomes "16". "Times" tells us to multiply, so it is replaced by an "x". "The sum of" says to add some numbers together. "Pi" and "–4.5" are the numbers we add. Putting all this together, we obtain, "A = 16 x [π + (–4.5)]", an equation that can be fed to a calculator! By properly entering the numbers and operations, we calculate an answer, A = –21.7.

Translators who are efficient know all the words of both languages. Beginners use a dictionary a lot. We have included a table which translates words from English into the equivalent mathematical form. It will be of use to you as you develop proficiency. Keep in mind that the goal is to successfully translate the *meaning* of the English sentence into the equivalent mathematical equation. Rigid transliteration of words using this "dictionary" can cause problems if you don't think carefully about what you are doing.

Stated problems should not be worked on the first reading. The first time you read a stated problem, you should answer the questions, "What type of stated problem is it?", "What is the unknown?" and "Which numbers given are important?" We are presently dealing with translation problems, but there are many other kinds dealt with in this chapter.

English Word	Mathematical Equivalent	Remarks/Examples
and	+	
from, diminished by	–	
of, times	x	
per	÷, /	"Percent" means "per 100" or /100

what, how much, how many...	A	A variable to solve
A "to be" verb (is, was, should be...) or "becomes"	=	

To show how translation works, we will now work a series of translation problems of increasing difficulty.

<u>Translation Example 1.</u> What is twenty-one percent of 349, diminished by 45?

The answer we seek is the result of a string of computations. We can translate this immediately. "What"→ A, "is" → =, "twenty-one percent" → 21/100 or 0.21, "of" → x, "diminished by" → –. Substituting,

$$A = 0.21x349 - 45 = 28.3 \quad .$$

<u>Translation Example 2.</u> The sum of two numbers is –110 and their difference is 38. What is their product?

Here the answer is the product of two numbers. We need to know the value of each number, so this becomes our goal. To commence translating, "The sum of" → "add what follows", "two numbers" → x,y, "is" → =. This gives

$$x + y = -110.$$

The second sentence is translated: "Their difference" →x – y, "is" → =. This produces

$$x - y = 38$$

The previous two mathematical equations are sufficient to solve for x and y. There are numerous ways to do this, and one will be described. We can add the left terms of each equation and set that equal to the sum of the right terms. From this x may be solved easily.

$$(x + y) + (x - y) = -110 + 38$$

$$x + x + y - y = -72$$

$$2x = -72 \qquad \text{or} \qquad x = -36$$

Now, y is obtained by substitution of –36 for x in either of the starting equations.

$$-36 + y = -110 \qquad \text{or} \qquad y = -74$$

At this point, it is a good idea to try our values of x and y in the second equation as a quick check. Is x – y equal to 38? –36 – (–74) does indeed equal 38, so we know we have solved for x and y correctly. We are not through though. We may now translate the last sentence. "What" → A, "is" → =, "their product" → xy.

$$A = xy = (-36)(-74) = 2660$$

Remember that, excepting "special" stated problems described in Section M of this chapter, the final answers on the Contest and in this manual are given with three significant digits, so the actual answer, 2664, is rounded to 2660 when entering it as an answer. A discussion of significant digits appears in Chapter 2D.

<u>Translation Example 3.</u> Jo is 5 years older than her brother. In 4 years she will be twice as old. How old is her brother?

The answer we seek is Jo's brother's age. We let this parameter equal B. For the first sentence, we translate "Jo" to mean "Jo's age" → J, "is" → =, "5 years" → 5, "older than" → +, "her brother" means "her brother's age" → B. This equation then becomes

$$J = 5 + B.$$

In 4 years, Jo's age will be J+4, and her brother's age will become B+4. The translation then is, "she" → J+4, "will be" → =, "twice as old" → 2x, "her brother's age(implied)" → B+4.

$$J + 4 = 2(B + 4)$$

Again, we have two equations and two unknowns, yielding J = 6 and B = 1. The answer then, to three significant digits, is B = 1.00.

Translation Example 4. The product of three consecutive integers is 30,958,830. What is their average? (integer)

This is a common trick problem on Calculator Applications Contests. The normal, technically correct approach to a solution is to set the three integers equal to m, n and p (or some other variable), and then to use the consecutive integer fact to write more equations like m = n–1, p = n+1. Substituting into the translated equation, we obtain

$$mnp = (n-1)(n)(n+1) = 30{,}958{,}830$$

$$n^3 - n = 30{,}958{,}830$$

This is a cubic equation, which could be solved using the calculator's Solver capability. There is an easier, and more importantly, faster, approach. When dealing with consecutive integers, we can figure that they are all almost equal to each other. In essence, we're saying that $m \approx n \approx p$! The ≈ sign means, "is almost equal to". Substituting this into the translated equation yields

$$mnp \approx n^3 \approx 30{,}958{,}830$$

$$n \approx (30{,}958{,}830)^{1/3} \approx 313.999$$

The answer may not be exactly the right answer, but it will usually be within ±1 of the correct answer. We determine the correct answer by substituting this into the original equation. That is, does (313)(314)(315) = 30,958,830? Yes! Now that we know what the variables are, we look to see what the question is, namely, what is the average of the three numbers.

$$A = \text{average}(313,314,315) = \frac{313 + 314 + 315}{3} = 314 \text{ (integer)}$$

D. Problems Involving Unit Conversions

Many stated problems require changing the units from one form to another. For example, we have a length in inches which must be changed to centimeters or miles; we need to change a number in grams to pounds, or seconds to weeks, or miles per hour to feet per second. We accomplish this by multiplying the number we want to change units on by "one", but this "one" is a special one symbolized in this Contest by "squiggly" brackets: {1}. Indeed, in this Contest Manual, the only time this type of bracket is used is around a unit conversion "one".

Most people know how to change a measurement in inches into feet. We know that 30 in. is 2.5 ft. The basis for this is that 12 in. = 1 ft. We can write this equality as

$$12 \text{ in} = 1 \text{ ft}$$

The units "in" and "ft" are really important here, because their absence leads to the nonsensical relation 12 = 1. Further, any unit is considered to be multiplied to the associated number. That is, "12 in" is mathematically considered to be "(12)(in)". Since both sides of the equation are non-zero, we can divide both sides by 12 in, to obtain a "conversion factor" {1}.

$$1 \equiv \{1\} = \left\{\frac{1 \text{ ft}}{12 \text{ in}}\right\}$$

The ≡ sign is used mathematically to imply an equality based on a definition. It is read, "is defined to be" rather than "equals". Since by our convention any fraction in squiggly brackets equals one, we may multiply or divide any term by this fraction without changing the term's value. To convert the 30 inches, we start with

$$A = 30 \text{ in} \quad .$$

Next we can multiply the right side by {1} in the form of the above expression.

$$A = 30 \text{ in} \left\{\frac{1 \text{ ft}}{12 \text{ in}}\right\} = (30)(\text{in}) \left\{\frac{(1)(\text{ft})}{(12)(\text{in})}\right\}$$

The unit "in" appears in the numerator and the denominator, so we can cancel them the same way we can cancel a common factor in a fraction. The result is

$$A = (30)(\cancel{\text{in}}) \left\{\frac{(1)(\text{ft})}{(12)(\cancel{\text{in}})}\right\} = (30)\left[\frac{(1)(\text{ft})}{12}\right] = \frac{30}{12} \text{ ft} = 2.5 \text{ ft} \quad .$$

The fun part about this approach is that we can string out conversion factors as far and as much as is needed without changing the value of the answer. Suppose we wanted to know how many seconds were in 1995. This is not a leap year, so there are 365 days in it. We could start multiplying numbers like 24 and 60 all over the place, but it is actually easier to guide the computation using conversion factors. We start with the value 1 yr.

$$A = 1 \text{ yr}$$

We want to change it to seconds eventually, but we don't know the direct conversion factor. We can get the answer quickly by stringing together conversion factors that we do know, making certain that the first one we write has a unit of "yr" in the denominator to cancel out the "yr" that's already there in the equation. We know that there's 365 days in 1995, a non-leap year, so we start with that. The two "yr" terms cancel and we have "dy" left over. We know there's 24 hr in a day, so that forms the next {1}, oriented so that the "dy" terms cancel. We continue with conversion factors for minutes and seconds as shown, and cancel units appearing in both the numerator and the denominator.

$$A = 1 \cancel{\text{yr}} \left\{\frac{365 \cancel{\text{dy}}}{1 \cancel{\text{yr}}}\right\} \left\{\frac{24 \cancel{\text{hr}}}{1 \cancel{\text{dy}}}\right\} \left\{\frac{60 \cancel{\text{min}}}{1 \cancel{\text{hr}}}\right\} \left\{\frac{60 \text{ sec}}{1 \cancel{\text{min}}}\right\} = (365)(24)(60)(60) \text{ sec}$$

$$A = 31{,}536{,}000 \text{ sec (integer)}$$

You may think that this is doing the problem the hard way, since we get the answer by multiplying one year by the larger number of the conversion factor. That's true here, but it's not always the case. Suppose we wanted to know the number of kilometers in a marathon, knowing that a marathon is 26.22 miles. The problem is easy if we know the conversion factor for miles and kilometers, but suppose we only remember the metric to English conversion factor, 1 in = 2.54 cm. The Calculator Applications Contest is not "open book", so the only conversion factors you can use are the ones you remember or what are permanently built in your calculator! Using the unit conversion approach described here, we string out conversion factors to change a mile into an inch, then we convert to centimeters, and finally we convert to kilometers. The string looks like this.

$$A = 26.22 \cancel{\text{mi}} \left\{\frac{5280 \cancel{\text{ft}}}{1 \cancel{\text{mi}}}\right\} \left\{\frac{12 \cancel{\text{in}}}{1 \cancel{\text{ft}}}\right\} \left\{\frac{2.54 \cancel{\text{cm}}}{1 \cancel{\text{in}}}\right\} \left\{\frac{1 \cancel{\text{m}}}{100 \cancel{\text{cm}}}\right\} \left\{\frac{1 \text{ km}}{1000 \cancel{\text{m}}}\right\}$$

$$A = \frac{(26.22)(5280)(12)(2.54)}{(100)(1000)} \text{ km} = 42.2 \text{ km}$$

Once you gain proficiency at working unit conversion problems, it no longer becomes necessary to write down the conversion factors. For the marathon problem, you would enter the "26.22" into the calculator and then multiply by 5280, keeping in mind that the number now has units of "ft". This continues until you finally get to "km" and the number on the display is the answer.

What follows are a few sample problems emphasizing unit conversions. Appendix A in this manual lists all the conversion factors you need to memorize to work unit conversions on tests. Any other conversion factors will be given explicitly in the problem statement.

Unit Conversion Example 1. In August, 1993, the U.S. dollar was worth 5.77 Finnish marks and also worth 1.66 German marks. What is the value of 8,500 German marks in Finnish marks? (FM)

$$A = 8500\ \cancel{GM} \left\{\frac{1\ \cancel{\$}}{1.66\ \cancel{GM}}\right\} \left\{\frac{5.77\ FM}{1\ \cancel{\$}}\right\} = 29{,}500\ FM$$

Unit Conversion Example 2. A car travels at 55 mph. What is the speed in units of megameters per year? (Mm/yr)

$$A = 55\ \frac{\cancel{mi}}{\cancel{hr}} \left\{\frac{5280\ \cancel{ft}}{1\ \cancel{mi}}\right\} \left\{\frac{12\ \cancel{in}}{1\ \cancel{ft}}\right\} \left\{\frac{2.54\ \cancel{cm}}{1\ \cancel{in}}\right\} \left\{\frac{1\ \cancel{m}}{100\ \cancel{cm}}\right\} \left\{\frac{1\ Mm}{10^6\ \cancel{m}}\right\} \left\{\frac{24\ \cancel{hr}}{1\ \cancel{dy}}\right\} \left\{\frac{365.256\ \cancel{dy}}{1\ yr}\right\}$$

$$A = 776\ \frac{Mm}{yr}$$

Unit Conversion Example 3. Pearl packs pints of peach preserves. She purchases three pecks of peaches. A peck is two gallons, and in the preserving process the peaches cook down by 20 percent. How many pints does Pearl package? (pt)

$$A = (1 - 0.2)(3\ \cancel{pecks}) \left\{\frac{2\ \cancel{gal}}{1\ \cancel{peck}}\right\} \left\{\frac{8\ pt}{1\ \cancel{gal}}\right\} = 38.4\ pt$$

Unit Conversion Example 4. Joe discovered that there are three blades of grass in every square centimeter of lawn. How many blades are there in a residential lot 150 ft by 100 ft if the house occupies 2080 sq. ft and the driveway is 40 ft long and 18 ft wide? (blades)

There is no grass growing in the area occupied by the house and driveway. These are subtracted from the lot total area.

$$A = [(150)(100) - 2080 - (40)(18)]\ \cancel{ft}^2 \left\{\frac{12\ \cancel{in}}{1\ \cancel{ft}}\right\}^2 \left\{\frac{2.54\ \cancel{cm}}{1\ \cancel{in}}\right\}^2 \left\{\frac{3\ blades}{1\ \cancel{cm^2}}\right\} = 3.40x10^7\ blades$$

In this problem, we squared the conversion factor, knowing that $1^2 = \{1\}^2 = 1$. The units inside the conversion factor get squared, too, so that

$$\left\{\frac{12\ in}{1\ ft}\right\}^2 = \left\{\frac{144\ in^2}{1\ ft^2}\right\} .$$

Unit Conversion Example 5. The Grand Coulee Dam on the Columbia River, Washington, was built from 1933 to 1941 and cost \$56 million. It weighs 19,285 tons and is built of concrete. Find the cost/cu.ft if concrete weighs 100 pounds/cu.ft. (¢/cu.ft)

$$A = \frac{\$56{,}000{,}000}{19{,}285\cancel{tons}} \left\{\frac{100¢}{1\$}\right\} \left\{\frac{1\cancel{ton}}{2000\cancel{lb}}\right\} \left\{\frac{100\cancel{lb}}{1ft^3}\right\} = 14{,}500\ \frac{¢}{ft^3}$$

In this problem, we look at the unknown, ¢/ft^3, and determine that a cost per unit volume is needed. With the density provided in the problem statement, we can get the right form of the answer as a cost per unit weight. We start off with the cost divided by weight, and then string conversion factors to get us to the right units on the answer.

Unit Conversion Example 6. The final draft of a report has 185 single-spaced pages. Six thousand double-sided copies are needed. How many 500-sheet reams of copy paper are needed? (integer,rms)

This is an integer problem, one of the "special" stated problems described in Section Mi of this chapter. We know this because the word, "integer", appears in the answer blank. Integer problems must be answered to the "ones" digit, independent of the number of significant digits. The number N of double-sided (i.e., copied on the front and back like this manual) pages in one report is

$$N = 1 \cancel{\text{report}} \left\{\frac{185 \cancel{\text{SSPages}}}{1 \cancel{\text{report}}}\right\} \left\{\frac{1 \text{ DSPage}}{2 \cancel{\text{SSPages}}}\right\} = 92.5 \text{ DSPages}$$

The back side of the last page of each report is blank, since it is not possible to have a fraction of a sheet of paper. This means that each report requires 93 sheets of paper.

$$A = 6000 \cancel{\text{reports}} \left\{\frac{93 \cancel{\text{Sheets}}}{1 \cancel{\text{report}}}\right\} \left\{\frac{1 \text{ ream}}{500 \cancel{\text{Sheets}}}\right\} = 1116 \text{ reams}$$

E. Problems Involving Rates

i. The Rate Equation

A rate in its strictest sense refers to anything that changes continuously with time. The most common rate is a velocity or speed r, the change in distance d with time t. We may easily write the "rate equation" as

$$d = rt$$

A rate might just as well be a texter's typing speed (words/min), a pitchers' throwing speed (mph), how fast a worker can weld (in/min), how fast a painter can paint an area (ft^2/hr), the flow rate of water through the Straits of Gibraltar (km^3/s), how fast a candle is consumed (in^3/hr) or how fast wheels on a bicycle rotate (RPM). The product of this rate r and time t gives a measure of the quantity that is changing.

Many are familiar with the rate equation as it describes distance d traveled in time t by a vehicle moving at a velocity r. Lots of stated problems on the Calculator Applications Contest involve problems of motion of cars, trucks, planes, rocket ships, trains, light, runners, walkers, etc. Some of these rate problems can be very difficult, involving several moving objects with interrelated time or distance. In all cases, the solution is found by using the rate equation for each moving or changing object. The following three problems, with different levels of difficulty, demonstrate the fundamental concept of the rate equation for moving objects. The last problem illustrates how one must be careful when dealing with average velocity.

Rate Example 1. I live 1.25 miles from my office and walk back and forth. If my step is 2.5 ft and I take 20 steps in 8 seconds, how long does it take me to walk home? (min)

This is a rate problem which gives us an opportunity to use conversion factors from the previous section on Unit Conversions. The rate r is in its most basic form 20 steps per 8 seconds. The question, "How long?" tells us that a time t is the unknown. Solving the rate equation for time, substituting for rate and distance d = 1.25 mi, and converting, we obtain the answer.

$$t = \frac{d}{r} = \frac{1.25\text{ mi}}{\dfrac{20\text{ steps}}{8\text{ s}}} = \frac{8(1.25\ \cancel{\text{mi}})\ \cancel{\text{s}}}{20\ \cancel{\text{steps}}} \left\{\frac{1\ \cancel{\text{step}}}{2.5\ \cancel{\text{ft}}}\right\} \left\{\frac{5280\ \cancel{\text{ft}}}{1\ \cancel{\text{mi}}}\right\} \left\{\frac{1\text{ min}}{60\ \cancel{\text{s}}}\right\} = 17.6\text{ min}$$

Rate Example 2. On a pitching game, targets move right to left with a velocity of 2 ft/s. Balls are thrown from 30 ft at the targets. If a person throws a ball at 45 mph, how far ahead of a target should she aim? (in)

We have two moving objects in this problem, the target and the ball. They move with different velocities and travel different distances, but the time is common to both. In these kinds of problems involving several moving objects, it is often most clear to make a table of three columns (rate, time, distance) with one row for each moving object.

A is the answer, the distance the target travels while the ball is in the air. The times are equal, since the time "origin" for the ball is the point at which the thrower lets loose the ball. An approach to the solution is to solve the ball equation for time, substitute that into the target equation and solve for A, knowing that $t_b = t_t$. The ball's traveling time is

Object	Rate	Time	= Distance
Target	2 ft/s	t_t	= A
Ball	45 mph	t_b	= 30 ft

$$t_b = \frac{30\text{ ft}}{45\text{ mi/hr}} =$$

$$\frac{(30\ \cancel{\text{ft}})\ \cancel{\text{hr}}}{45\ \cancel{\text{mi}}} \left\{\frac{1\ \cancel{\text{mi}}}{5280\ \cancel{\text{ft}}}\right\} \left\{\frac{60\ \cancel{\text{min}}}{1\ \cancel{\text{hr}}}\right\} \left\{\frac{60\text{ s}}{1\ \cancel{\text{min}}}\right\} =$$

$$0.455\text{ s} \quad .$$

Substituting this time for the target travel time in the target's rate equation, we obtain the answer, the distance the target travels while the ball is in the air.

$$A = \frac{2\ \cancel{\text{ft}}}{\cancel{\text{s}}}\ (0.455\ \cancel{\text{s}}) \left\{\frac{12\text{ in}}{1\ \cancel{\text{ft}}}\right\} = 10.9\text{ in}$$

Rate Example 3. A student drives from Austin to Waco to watch a basketball game, averaging 63 mph. His average speed going and coming was 52 mph because it was raining during the return trip. What was his average speed on the return trip? (mph)

The quick (and incorrect!) way to work this problem is to assume that the velocity going to Waco and the velocity returning from Waco may be averaged to obtain the average velocity. Using this approach, an answer of 41 mph is quickly obtained. The problem with this approach is that the average velocity is the velocity averaged over time, and when the student drives fast, he covers the distance in less time than when he drives more slowly. This factor weights the slower velocity, so the answer to this example will in actuality be higher than 41 mph.

The general solution is derived here. Let the velocity going to Waco be r_1. The returning velocity is r_2. The average velocity r_{avg} is the total distance traveled going ($d_1 = r_1 t_1$) and coming ($d_2 = r_2 t_2$) divided by the total elapsed time ($t_1 + t_2$).

$$r_{avg} = \frac{d_1 + d_2}{t_1 + t_2} = \frac{d_1 + d_2}{\left[\dfrac{d_1}{r_1} + \dfrac{d_2}{r_2}\right]}$$

Since the distance from Austin to Waco (d_1) equals the distance from Waco to Austin (d_2), the distance values in the equation all cancel, yielding the final equation.

$$r_{avg} = \frac{2}{\left[\frac{1}{r_1} + \frac{1}{r_2}\right]} \quad \text{or} \quad \frac{1}{r_{avg}} = \frac{\left[\frac{1}{r_1} + \frac{1}{r_2}\right]}{2}$$

From this last form, it is seen that the proper quantity to average is the <u>reciprocal</u> of velocity, not the velocity itself. Substituting the numbers from the problem statement, we can now solve for the return velocity r_2.

$$\frac{1}{r_{avg}} = \frac{\left[\frac{1}{r_1} + \frac{1}{r_2}\right]}{2} \quad \text{or} \quad \frac{1}{52\text{ mph}} = \frac{\left[\frac{1}{63\text{ mph}} + \frac{1}{r_2}\right]}{2}$$

$$\frac{1}{r_2} = \frac{2}{52\text{ mph}} - \frac{1}{63\text{ mph}} = \left[\frac{2}{52} - \frac{1}{63}\right]\text{ mph}^{-1} \quad \text{or} \quad r_2 = \frac{1}{\left[\frac{2}{52} - \frac{1}{63}\right]}\text{ mph} = 44.3\text{ mph}$$

Rate problems can also deal with things besides distance. The key here is to remind yourself that something is getting done at some rate, to start with the rate equation and to trust conversion factors to make everything come out okay in the end! These sample problems show how the conventional distance d can be unconventional: water volume in a fountain, the level of the Mediterranean Sea, a load of logs and a 354-page book!

<u>Rate Example 4.</u> My fountain fills in 4 hours with a garden hose. However, it takes 6.5 hours to drain by siphoning. How long would it take to fill if the siphon were operating? (hr)

We do well here to build a rate table with one row for filling, one row for siphoning, and one row for filling while siphoning. From these relations, we can calculate the rate of filling r_f and draining r_d to be

Task	Rate	Time	= Distance
Filling	r_f	4 hr	= 1 Fountain
Siphoning	r_d	6.5 hr	= 1 Fountain
Filling *and* Siphoning	$r_f - r_d$	A	= 1 Fountain

$$r_f = \frac{1\text{ Fountain}}{4\text{ hr}} \qquad r_d = \frac{1\text{ Fountain}}{6.5\text{ hr}} \quad .$$

Assuming that draining and filling are independent operations, the net rate of simultaneous filling and draining is simply $r_{net} = r_f - r_d$. The time to fill under these circumstances is the answer, A = 1 Fountain/r_{net}.

$$A = \frac{1\text{ Fountain}}{\left[\frac{1\text{ Fountain}}{4\text{ hr}} - \frac{1\text{ Fountain}}{6.5\text{ hr}}\right]} = \frac{1\ \cancel{\text{Fountain}}}{\left[\frac{1}{4} - \frac{1}{6.5}\right]\frac{\cancel{\text{Fountain}}}{\text{hr}}} = 10.4\text{ hr}$$

<u>Rate Example 5.</u> The Mediterranean Sea covers 2,510,000 km^2. How long would it take to lower the Sea's level by 1 inch if it drains through the Straits of Gibraltar at a rate of 25 mi^3/hr? (min)

The rate here is a volume flow rate, r_v. The volume of material V which flows equals the volume flow rate times time t. Solving this equation for t yields our answer.

$$t = t = \frac{V}{r_v} = \frac{\left[(2{,}510{,}000\cancel{\text{km}}^2)(1\cancel{\text{in}})\right]}{25\cancel{\text{mi}}^3/\cancel{\text{hr}}}\left\{\frac{60\text{min}}{1\cancel{\text{hr}}}\right\}\left\{\frac{1\cancel{\text{mi}}}{5280\cancel{\text{ft}}}\right\}^3\left\{\frac{1\cancel{\text{ft}}}{12\cancel{\text{in}}}\right\}^3\left\{\frac{1\cancel{\text{in}}}{2.54\cancel{\text{cm}}}\right\}^2\left\{\frac{100\cancel{\text{cm}}}{1\cancel{\text{m}}}\right\}^2\left\{\frac{1000\cancel{\text{m}}}{1\cancel{\text{km}}}\right\}^2$$

$$t = 36.7 \text{ min}$$

Rate Example 6. One person can haul a load of logs by hand in 12 hours. Two people do the job more efficiently, finishing in 5 hours. If John starts a load and 4.3 hours later is joined by Jim, how much longer does it take to finish the job? (hr)

Task	"Rate"	Time	= "Distance"
1 Person	r_1	12 hr	= 1 LogLoad
2 Persons	r_2	5 hr	= 1 LogLoad
John Alone	r_1	4.3 hr	= d_{John}
John and Jim	r_2	A	= 1 LogLoad – d_{John}

As before, we make a table and solve for the unknown time, with our "distance" value equal to 1 "LogLoad".

As in the earlier examples, each row is an equation. We need as many equations as unknowns to solve for the unknowns. There are four unknowns: r_1, r_2, d_{John} and A. To move as directly as possible to the answer A, we work back from the last-row equation. r_2 and d_{John} are needed though. The first variable can be obtained from the second-row equation.

$$r_2 = \frac{1}{5} \frac{\text{LogLoad}}{\text{hr}}$$

We get the second variable from the row-three equation, but we see that we need r_1 first. It comes from the first-row equation.

$$r_1 = \frac{1}{12} \frac{\text{LogLoad}}{\text{hr}} \quad \text{, so}$$

$$d_{John} = (4.3 \cancel{\text{hr}}) \left[\frac{1}{12} \frac{\text{LogLoad}}{\cancel{\text{hr}}}\right] = \frac{4.3}{12} \quad \text{Logload}$$

Now we can calculate the answer using the equation from the last row.

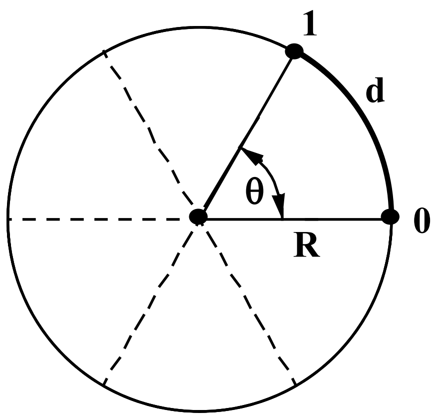
$$A = \frac{1 \text{ LogLoad} - d_{John}}{r_2} = \frac{1 \text{ LogLoad} - \frac{4.3}{12} \text{ Logload}}{\frac{1}{5} \frac{\text{LogLoad}}{\text{hr}}} = \frac{\left[1 - \frac{4.3}{12}\right] \cancel{\text{Logload}}}{\frac{1}{5} \frac{\cancel{\text{LogLoad}}}{\text{hr}}} = 5\left[1 - \frac{4.3}{12}\right] \text{ hr}$$

$$A = 3.21 \text{ hr}$$

Rate Example 7. Molly proofreads 125 words per minute and Sue proofreads 119. To proofread a 354 page book in the shortest time, how many pages does Sue proofread? (integer)

Person	"Rate"	Time	= "Distance"
Molly	125 word/min	t_m	= d_m
Sue	119 word/min	t_s	= d_s = A
Molly *and* Sue	(125 + 119) word/min	t_{total}	= 354 pages

The rate table is shown. One potential problem is that we do not know how many words there are on a page. This turns out not to be a problem, because the word-to-page ratio scales and the scaling factor drops out eventually.



All three times are equal, since Molly and Sue begin and end at the same time. We replace the time variables with "t". Further, assume on average that there are W words on each page. The first two equations are shown.

$$d_m = \frac{125 \cancel{\text{word}}}{\text{min}}\ t \left\{\frac{1 \text{ page}}{W \cancel{\text{word}}}\right\} = \left[\frac{125}{W}\right]\left[\frac{t}{\text{min}}\right] \text{ pages} \quad , \quad d_s = \left[\frac{119}{W}\right]\left[\frac{t}{\text{min}}\right] \text{ pages}$$

By writing a ratio between these values, we find that the unknown time and the unknown W cancel.

$$\frac{d_m}{d_s} = \frac{\left[\frac{125}{\cancel{W}}\right]\left[\frac{\cancel{t}}{\cancel{\text{min}}}\right] \cancel{\text{pages}}}{\left[\frac{119}{\cancel{W}}\right]\left[\frac{\cancel{t}}{\cancel{\text{min}}}\right] \cancel{\text{pages}}} = \frac{125}{119} \quad \text{or} \quad d_m = \frac{125}{119}\ d_s$$

The sum of d_m and d_s is 354 pages. Substituting the above equation yields

$$d_m + d_s = 354 \text{ pages} = \frac{125}{119} d_s + d_s$$

$$d_s = \frac{354 \text{ pages}}{\left[\frac{125}{119} + 1\right]} = 173 \text{ pages}$$

Rotational motion is also an important aspect of many real-life experiences. Examples are clock hands, merry-go-rounds, planetary orbits, ferris wheels, bicycle and car tires, records and CDs, and circular saw blades. There are also scientific and engineering applications for rotational motion, such as inertial or gyroscopic devices and alternating current. First, consider how we measure angles. In the figure, we show an angle labeled "θ", as marked in a circle of radius R. This defines an arc length d. In the Calculator Applications Contest, we use three ways to measure this angle: revolutions, degrees and radians. In revolution measure, we use a "piece of pie" analogy. The circle may be taken to be a pie, and the fraction of the entire pie represented by the angular "slice" represents the angle in units of revolutions. In the figure, the slice is one sixth of the total pie, so our angle measure is

$$\theta = \frac{1}{6} \text{ rev} \approx 0.167 \text{ rev} \quad .$$

The most common unit for measuring angles is degrees. The circle is split into 360 radial parts or slices, each part representing an angle of 1 degree. By normal convention, degrees may be expressed in decimal fraction (e.g., 35.64°), or degrees may be split into 60 subparts called minutes, and each minute may be split into 60 subparts called a second. The symbol for a minute of an angle is a single quote ', and a second of arc is represented by a double quote ". In this measure, 35.64° equals 35°38'24". It may be confusing at first to think of minutes and seconds as fractions of an angle rather than a unit of time. Also, many calculators have a key which converts hours/minutes/seconds into decimal hours. This key may also be used to quickly convert between degree measure in degree/minute/second and decimal notation. We can build a conversion factor to relate degree measure to revolution measure. One revolution equals 360°. We can use the figure to find the angle θ in degree measure.

$$\theta = \left(\frac{1}{6} \cancel{\text{rev}}\right)\left\{\frac{360°}{\cancel{\text{rev}}}\right\} = 60°$$

Mathematically the most "natural" way to measure an angle is radian measure. The circumference of a circle is $2\pi R$. We can measure an angle as the arc length d (see figure) divided by the radius of the circle. One revolution or 360° is equal to the circumference $2\pi R$ divided by R, or 2π radians. If 360° equals 2π radians, then 90° is $\pi/2$ radians, 180° is π radians, etc. The angle in the figure is written in radian measure as

$$\theta = \left(\frac{1}{6}\,\cancel{\text{rev}}\right)\left\{\frac{2\pi\ \text{rad}}{\cancel{\text{rev}}}\right\} = \frac{\pi}{3}\ \text{rad} \approx 1.05\ \text{rad}$$

The length of arc d in the figure may be written in the following way for each method of the angular measure. In all cases, it is given by a linear proportionality between arc length d and the associated angle θ. The constant of proportionality is k.

$$d = k\theta \qquad \text{or} \qquad \frac{d_2}{d_1} = \frac{k\theta_2}{k\theta_1} = \frac{\theta_2}{\theta_1} \qquad \text{or} \qquad d_2 = \left[\frac{d_1}{\theta_1}\right]\theta_2$$

We can substitute for d_1 and θ_1 the values for a complete circle, $d_1 = 2\pi R$ and $\theta_1 = 1$ rev, 360° or 2π radians. Also, we can drop the "2" subscript on the remaining variables to obtain a general equation.

$$d = \left[\frac{2\pi R}{1\ \text{rev}}\right]\theta = 2\pi R\left[\frac{\theta}{1\ \text{rev}}\right] = 2\pi R\left[\frac{\theta}{360°}\right] = 2\pi R\left[\frac{\theta}{2\pi\ \text{rad}}\right]$$

When this equation is written in radian measure, it has common factors. Following the usual convention in science and engineering, the unit "rad" is moved to the radius variable and absorbed to make a pretty equation with odd-ball units on R. We note this unit change in the variable by a bold character **R**. The only difference between R and **R** is the units; R has units of length (in, km) whereas **R** has units of length/radian (in/rad. km/rad).

$$d = \cancel{2\pi}\,R\left[\frac{\theta}{\cancel{2\pi}\ \text{rad}}\right] = \left[\frac{R}{\text{rad}}\right]\theta = \mathbf{R}\theta$$

This is illustrated in Rate Example Problem 8.

Now that the three ways we measure angles have been described, we can deal with rotational motion. Generally, we think of a point moving counterclockwise around a circle of radius R. We start timing the motion when the point is at Point 0 on the figure (i.e., the angle θ equals 0 by any measurement scheme at Point 0). As the angle θ increases linearly with time t, we can define an angular velocity ω as the rate at which the angle increases.

$$\omega = \frac{\theta}{t} \qquad \text{or} \qquad \theta = \omega t$$

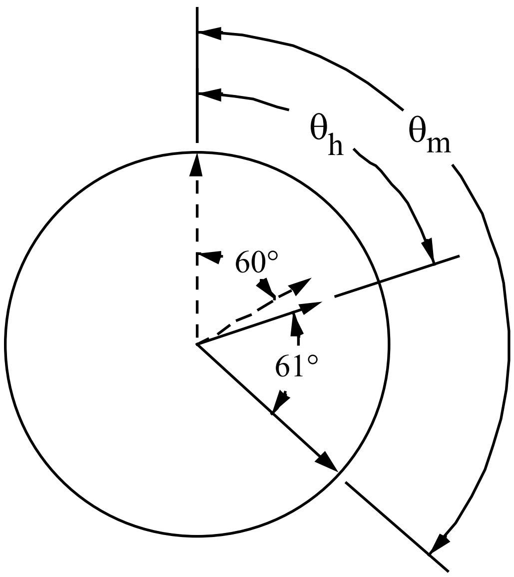
The units of ω are usually revolutions per minute (RPM) or radians per second. The rate r of the point moving around the circle is technically the arc length d traversed in a time t. We can obtain a relationship between r and ω if the angular measure is radians.

$$d = \mathbf{R}\theta = rt = \mathbf{R}(\omega t) = (\mathbf{R}\omega)t \qquad \text{or} \qquad r = \mathbf{R}\omega$$

<u>Rate Example 8.</u> A Weedeater™ that turns 6000 rpm has a cutting cord that swings in a circle that is 12 inches in diameter. What is the speed of the tip of the cord? (in/sec)

We use the relation $r = \mathbf{R}\omega$ and solve directly.

$$r = \mathbf{R}\omega = \left[6\,\frac{\text{in}}{\cancel{\text{rad}}}\right]\left[6000\,\frac{\cancel{\text{rev}}}{\cancel{\text{min}}}\right]\left\{\frac{2\pi\ \cancel{\text{rad}}}{\cancel{\text{rev}}}\right\}\left\{\frac{1\ \cancel{\text{min}}}{60\ \text{s}}\right\} = 3770\,\frac{\text{in}}{\text{s}}$$



The types of problems dealing with rotational motion are varied. Examples are automobile tires, ferris wheels, merry-go-rounds, CDs and DVDs in players, ants crawling around a fountain, the rotation of planets, spinning tops, lawn mower blades, a looped lariat, ice skaters in spins, satellites in orbit, an object in a tornado, a hurricane, bicycle spokes. One of the most common problem types for our contest dealing with the rotational motion of the second, minute and hour hands of non-digital timepieces. The following example illustrates this type of problem.

Rate Example 9. How many minutes after 2 o'clock do the hour and minute hands of a clock form an angle of 61°? (min)

The geometry of the problem is shown in the figure. The key to this problem is to realize that a minute hand on a clock makes one revolution every hour (ω_m = 360°/hr), and the hour hand makes a revolution every 12 hours (ω_h = 360°/12hr = 30°/hr). Also, we need to know the relative position of the minute and hour hands at some point in time. At 2 O'clock, the minute hand points straight up and the hour hand points at the "2". The angular measure between the minute and hour hands θ_0 is

$$\theta_0 = \frac{2}{12}(360°) = 60° \ .$$

The angular distance between the minute and hour hand is $\theta_m - \theta_h$. Using vertically up as $\theta = 0°$ and clockwise as the direction of positive angle increase, we can write the following equations.

$$\theta_m = \omega_m t = \frac{360t°}{hr} \quad \text{and} \quad \theta_h = \omega_h t + \theta_0 = \frac{30t°}{hr} + 60°$$

$$\theta_m - \theta_h = \frac{360t°}{hr} - \left[\frac{30t°}{hr} + 60°\right] = \frac{330t°}{hr} - 60°$$

We seek the elapsed time t at which the difference between the hour and minute hands is 61°. The last equation gives this quantity, which may be solved and converted to minutes to obtain the answer.

$$\theta_m - \theta_h = \frac{330t°}{hr} - 60° = 61°$$

$$\frac{330t°}{hr} = 121° \quad \text{or} \quad t = \frac{121°}{330°}\,hr = 0.367\,\cancel{hr}\left\{\frac{60\ min}{1\ \cancel{hr}}\right\} = 22.0\ min$$

ii. Acceleration Problems

The rate equations of the previous section are based on a constant velocity or rate. The equivalent distance then is the product of that rate and time. Acceleration problems assume that an object is accelerating. The velocity then changes with time as does the distance. Problems amenable to this style include gravity/falling objects and accelerating/decelerating vehicles.

Acceleration is given by the constant a. It is the time rate of change of velocity v and may be measured in units like mph/s, ft/s^2, or m/s^2. When an object accelerates from rest, and distance d and time t are measured from the start of the acceleration, the velocity and distance equations are given by:

$$v = at \quad \text{and} \quad d = \frac{1}{2}at^2$$

When an object is dropped, it will freefall in the earth's gravitational field. It actually accelerates. Objects accelerating in a gravitational field are subjected to a constant acceleration called the acceleration due to gravity, denoted by the variable g instead of a. The value of g is -9.807 m/s^2 = -32.17 ft/s^2, a "must know" constant which contestants must memorize. The negative sign is consistent with the usual convention that "up" is positive, and "down" is negative. Freefalling objects don't accelerate forever, because the air resistance balances the force of acceleration. We refer to the highest freefall speed as a "terminal velocity", which is around 120 mph. The traditional distance convention in gravity problems is that distance is measured from the ground, with "up into the air" being a positive distance measure and "down toward the ground" being negative.

Imagine a slow car driving down the road at 30 mph. Fifty feet after an intersection, a friend in a fast car passes at a constant velocity of 40 mph, and, after a time delay of 2 seconds, the slow car accelerates at a rate of 2 mph/second to catch up with the friend. We can describe at any point during the acceleration the slow car's velocity and distance. We will need to decide beforehand from what point distance is measured and from what point time is measured. Imagine distance as a tape measure, where the starting distance is measured on the tape measure at a value d_o and the "zero" end of the tape measure is some convenient location. For this problem, we could zero the tape measure at the location where the fast car passed the slow car, or we could zero the tape measure at the intersection. For the latter, the first car's initial position d_o is +50 ft. Next, let's consider time. One could imagine the problem is monitored by someone with a stopwatch that initially reads zero. The question is effectively, "When does the stopwatch start?" For this problem, we can set the "go" on the stopwatch at the instant the fast car passes the first car. The acceleration of the slow car starts after a time delay of 2 seconds, so the acceleration starts when the stopwatch reads 2 seconds. This imaginary stopwatch reading when acceleration starts is called t_o and for the first car t_o equals +2 seconds. We are now able to write the velocity and distance equations for the slow car.

The most general, standard forms of the velocity and distance equations are:

$$v = v_o + a(t - t_o) \quad \text{and} \quad d = d_o + v_o\left(t - t_o\right) + \frac{1}{2}a(t - t_o)^2$$

where a = constant acceleration, v = velocity, d = distance, t = time, v_o and d_o are associated values at which the acceleration initiates, and t_o is the "stopwatch" time at which acceleration commences. For our car problem, these equations would look like this:

$$v_s = 30\text{mph} + 2\text{mph/s}(t - 2\text{s}) \;\text{ and }\; d_s = 50\text{ft} + 30\text{mph}\left(t - 2\text{s}\right) + \frac{1}{2}(2\text{mph/s})(t - 2\text{s})^2$$

These equations give velocity and distance as a function of time, where time is measured from the instant when the fast car passed the slow car. It is apparent that some unit conversions will need to be made, particularly in the distance equation where ft and miles are used.

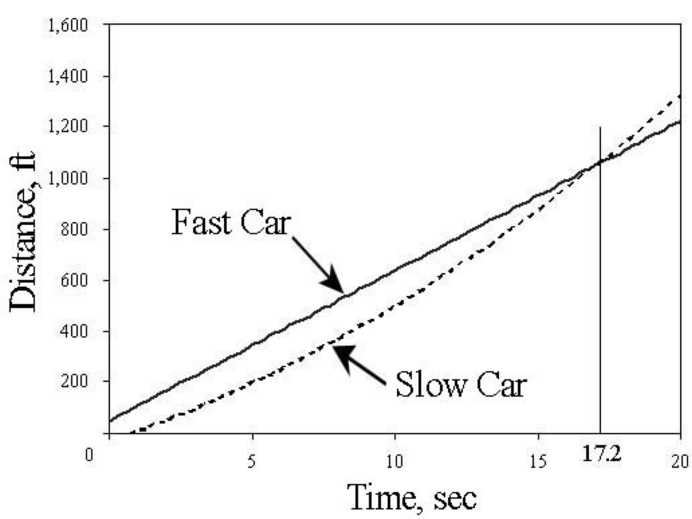
The fast car's velocity and distance equation are given by:

$v_f = 40\text{mph}$ and $d_f = 50\text{ft} + 40\text{mph}(t)$.

Setting the distance of the slow car equal to the distance of the fast car, we can calculate the time t required for the slow car to catch up to the fast car, where the time is elapsed from the point the fast car passes the slow car. This time calculates out to be 17.2 s. This is shown in the figure. The solution requires solving a quadratic equation for t, and the second root is always negative (-3.25 s for this problem) and should be disregarded since it does not make physical sense. The slow car's velocity at the meeting point is 60.4 mph. The cars are 1060 ft from the intersection when they meet.

The choice of the zero location in distance and time is arbitrary. They can be any value, and the correct answer will be calculated so long as the problem is worked correctly. Usually, the choice is made to make working the problem as easy as possible. For that reason, we are usually motivated to set the distance zero and the time zero at the start of the acceleration. This makes d_o and t_o both zero. When in addition, the initial velocity is zero, then t_o, v_o and d_o are all zero and the standard equations simplify to the more common forms:

$$v = at \quad \text{and} \quad d = \frac{1}{2}at^2$$



Further, for these simplified forms, we can write two more forms of the distance equation by substituting v = at:

$$d = \frac{1}{2}(at)t = \frac{1}{2}vt \quad \text{and}$$

$$d = \frac{1}{2}at^2\left(\frac{a}{a}\right) = \frac{1}{2}\frac{(at)^2}{a} = \frac{1}{2}\frac{v^2}{a}$$

These equations streamline solutions to certain problems.

Acceleration Example 1. A car can accelerate from 0 to 50 mph in 3.5 seconds. How far does it travel during the acceleration? (ft)

The answer comes directly as

$$d = \frac{1}{2}vt = \frac{1}{2}\left(50\frac{\text{mi}}{\text{hr}}\right)(3.5\,\text{sec})\left\{\frac{5280\text{ft}}{\text{mi}}\right\}\left\{\frac{\text{hr}}{3600\,\text{sec}}\right\} = 128\text{ft}$$

Acceleration Example 2. An amusement park free-fall ride advertises a 4 second free fall. If this comprises 70% of the total ride height, how tall is the ride? (ft)

Taking d_o and t_o both equal to zero, the solution comes from the simplified form of the acceleration equation is

$$d = \frac{1}{2}at^2 = \frac{1}{2}\left(-32.17\text{ft}/\text{s}^2\right)(4\text{s})^2 = -257\text{ft}$$

This is the "down" distance (negative), so the "up" distance or height is the positive value, 257 ft. But this comprises 70% of the ride height, so our answer A is given by 0.7A = 237 ft, or A = 368 ft.

Acceleration Example 3. A commercial jet lands at a land speed of 300 mph and decelerates to 60 mph at a constant rate in 5 seconds. What is the acceleration in g's, if 1 g is the absolute value of the acceleration of gravity at the earth's surface? The answer is a positive number. (g)

We have v_o = 300 mph, v = 60 mph, and t = 5 s, and we can take t_o = 0. Using the general form of the velocity equation, we obtain the solution.

$$v = v_o + a(t - t_o) \text{ or } a = \frac{v - v_0}{t - t_0} = \left[\frac{60\text{mph} - 300\text{mph}}{5\text{s} - 0}\right]\left\{\frac{5280\text{ft}}{\text{mi}}\right\}\left\{\frac{1\text{hr}}{3600\text{s}}\right\} = -70.4\text{ft}/\text{s}^2$$

The answer should be in units of "g", with 1 g = -32.17 ft/s^2 for a deceleration. The deceleration then becomes

$$a = -70.4\text{ft}/\text{s}^2\left\{\frac{1\text{g}}{-32.17\text{ft}/\text{s}^2}\right\} = 2.19\text{g}$$

Acceleration Example 4. A ball is dropped from a 200 ft tall building. After 1 sec, a second ball is thrown downward such that both hit the ground simultaneously. What is the second ball's initial velocity if air resistance is neglected? (ft/s)

The solution begins with the first ball, whose distance equation is

$$d = 200 \text{ ft} + \frac{1}{2}(-32.17 \text{ ft}/\text{s}^2)t^2$$

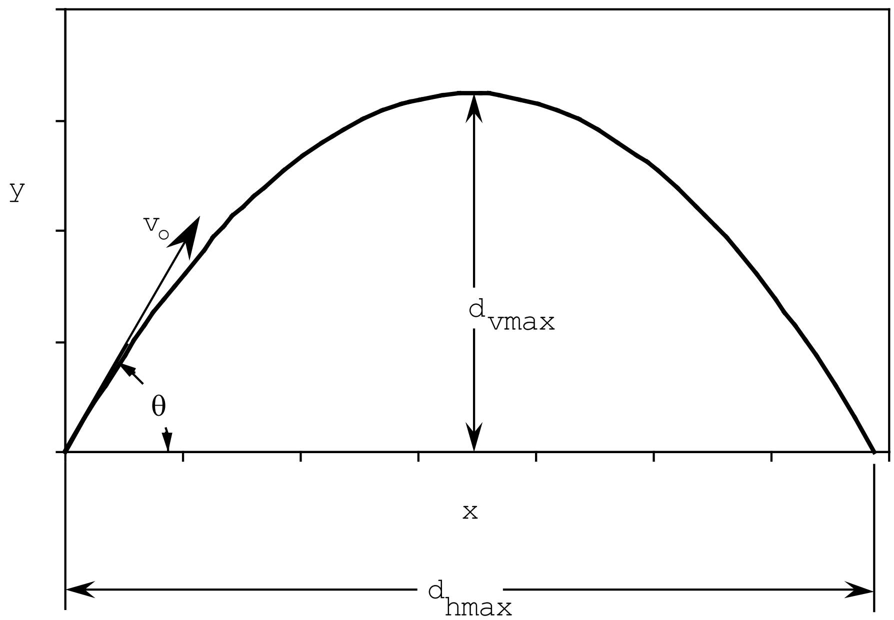
assuming that the problem stopwatch starts when it is dropped. Solving for the time to impact the ground (i.e., d = 0) we obtain t = 3.53 seconds. The second ball leaves at t_o = 1 s, so its equation of motion is

$$d = 200 \text{ ft} + v_o(t - 1 \text{ s}) + \frac{1}{2}(-32.17 \text{ ft}/\text{s}^2)(t - 1 \text{ s})^2.$$

Since both hit the ground at the same time, t = 3.53 s when d = 0 ft for the second ball. Solving for v_o, we obtain our answer, -38.5 ft/s.

iii. Trajectory Problems

Trajectories are everywhere: an egg at the holiday egg-toss contest, a rifle bullet, a golfer's golf ball, a human cannonball, the high-school quarterback throwing a bomb to a teammate. The list is pretty extensive. In all these instances, an object is "fired" or released at some initial velocity v_o and at some angle θ relative to the ground. It travels up and away, reaching a maximum vertical distance or height d_{vmax}. Then, the object falls back to the ground, at impact having traveled a maximum horizontal distance d_{hmax}. The figure shows the path or trajectory of such an object. During flight, the object is subjected to the earth's gravitational field which provides a downward (i.e., negative) acceleration g.



It turns out that the maximum vertical and horizontal travel of an object is predictable given the initial velocity and firing angle relative to the horizontal. The reverse is also true. If we have a maximum height and horizontal distance, then we can calculate the initial velocity and angle to obtain these values. The equations presented here are based on the acceleration equations described in the previous section. Further, assumptions are that the launch point is the origin with the launch occurring at time equal to zero, and the ending elevation is equal to the launch elevation. If v_o and θ are given, then the maximum horizontal and vertical distances are, respectively:

$$d_{h_{max}} = \frac{-v_o^2 \sin 2\theta}{g} \qquad \text{and} \qquad d_{v_{max}} = \frac{-v_o^2 \sin^2 \theta}{2g}$$

Given d_{hmax} and d_{vmax}, the required initial velocity and angle are given by:

$$v_o = \sqrt{\left(\frac{-g}{8d_{v_{max}}}\right)\left(d_{h_{max}}^2 + 16 d_{v_{max}}^2\right)} \qquad \text{and} \qquad \tan\theta = \frac{4d_{v_{max}}}{d_{h_{max}}}$$

The time of flight t_{of} of an object may be calculated, again assuming the landing elevation is equal to the launch elevation:

$$t_{of} = \frac{-2v_o \sin\theta}{g}$$

If the launch and ending vertical heights are not equal, a different set of equations must be used for the horizontal/vertical distances and time of flight. These equations are given following Trajectory Example 6.

Trajectory Example 1. A baseball player throws a ball from second base to home plate, 127 ft. The ball peaks at 45 ft above the release point. What is the initial release angle of the ball (degrees)?

We use the last equation for the angle.

$$\tan\theta = \frac{4d_{v_{max}}}{d_{h_{max}}}$$

The maximum vertical (45 ft) and horizontal (127 ft) distances are given, so we substitute and calculate the release angle: $\tan\theta = \frac{4(45\ \text{ft})}{127\ \text{ft}} = 1.417$ which yields $\theta = 54.8°$.

Trajectory Example 2. In soccer, a player kicks the ball at an angle of 35 degrees from the ground, and it travels horizontally 75 meters before bouncing. What was the ball's maximum vertical height? (m)

$$\tan\theta = \frac{4d_{v_{max}}}{d_{h_{max}}} \quad \text{or} \quad d_{v_{max}} = \frac{d_{h_{max}}\tan\theta}{4} = \frac{75\text{m}\tan(35^\circ)}{4} = 13.1\text{m}$$

Trajectory Example 3. The net for a human cannonball is 130 ft from the cannon. The angle of the cannon is 30° from horizontal. What is the human cannonball initial velocity? (mph)

We solve using the horizontal distance equation.

$$d_{h_{max}} = \frac{-v_o^2\sin 2\theta}{g} \quad \text{or} \quad v_0 = \sqrt{\frac{-d_{h_{max}}g}{\sin 2\theta}} = \sqrt{\frac{-(130\text{ft})(-32.17\text{ft/s}^2)}{\sin[2(30^\circ)]}} = 69.5\text{ft/s}\left\{\frac{\text{mi}}{5280\text{ft}}\right\}\left\{\frac{3600\text{s}}{1\text{hr}}\right\} = 47.4\text{mph}$$

Trajectory Example 4. In an egg toss game, Sam lobs an egg to his partner 20 ft away. The maximum height of the egg trajectory is 16 ft. What was the initial egg velocity? (mph)

$$v_o = \sqrt{\left(\frac{-g}{8d_{v_{max}}}\right)\left(d_{h_{max}}^2 + 16d_{v_{max}}^2\right)} = \sqrt{\left(\frac{-(-32.17\text{ft/s}^2)}{8(16\text{ft})}\right)\left[(20\text{ft})^2 + 16(16\text{ft})^2\right]}$$

$$v_0 = 33.6\text{ft/s}\left\{\frac{1\text{mi}}{5280\text{ft}}\right\}\left\{\frac{3600\text{s}}{1\text{hr}}\right\} = 22.9\text{mph}$$

Trajectory Example 5. Joey lobs a water balloon with an initial velocity of 20 mph at a release angle of 48° relative to the ground. To what maximum height above the release point does the balloon travel? (ft)

We use the maximum vertical distance equation.

$$d_{v_{max}} = \frac{-v_o^2\sin^2\theta}{2g} = \frac{-(20\text{mph})^2\sin^2(48^\circ)}{2(-32.17\text{ft/s}^2)}\left\{\frac{5280\text{ft}}{1\text{mi}}\right\}^2\left\{\frac{1\text{hr}}{3600\text{s}}\right\}^2 = 7.39\text{ft}$$

Trajectory Example 6. A bullet is fired at 300 ft/sec. What is the angle of the gun relative to the ground if the bullet travels a horizontal distance of 0.4 miles and the angle is less than π/4 radians? (rad)

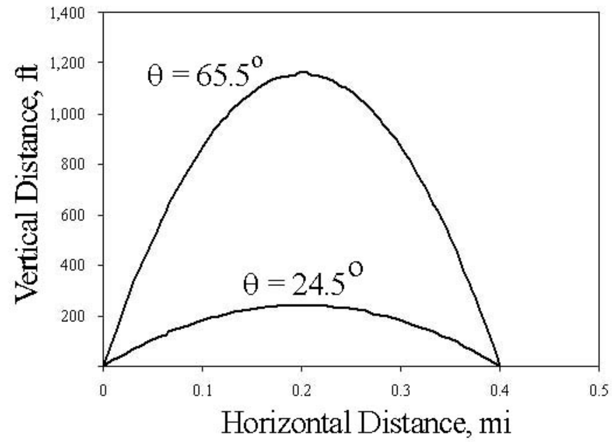
We start with the horizontal distance equation and solve for sin(2θ):

$$d_{h_{max}} = \frac{-v_o^2\sin 2\theta}{g}$$

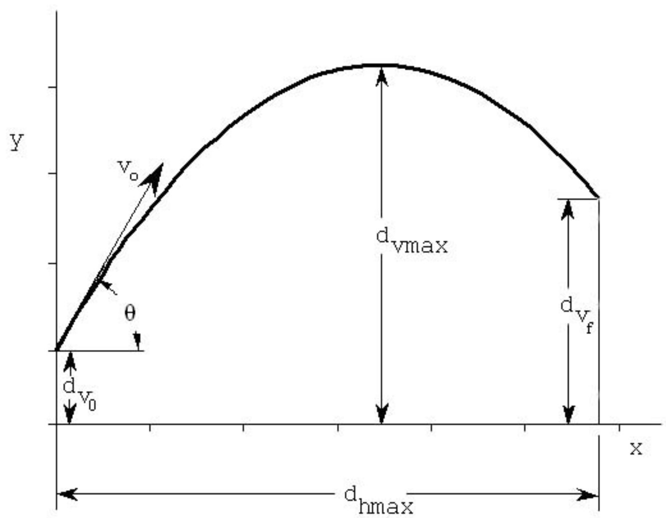
from which

$$\sin(2\theta) = \frac{d_{h\,max}\,g}{-v_o^2} = \frac{(0.4\text{mi})(-32.17\text{ft/s}^2)}{-(300\text{ft/s})^2}\left\{\frac{5280\text{ft}}{1\text{mi}}\right\} = 0.755$$

From this we solve for θ = 0.428 rad, about 24.5º. This brings up an issue associated with solving for launch angles. For a given maximum horizontal distance and initial velocity, there are two launch angles. One is greater than 45º and is akin to lobbing the object. The other is less than 45º (the case for the previous problem) and is associated with "clothes-lining" or "scudding" the object along the ground. This comes from the fact that sin(2θ) = sin(180º-2θ). This means for our analysis here that if θ is a solution, then 90º - θ is also a solution, θ = 65.5º.



When the ending elevation is *different* from the launch elevation, the equations described for d_{hmax} and d_{vmax} are not valid and new relationships must be developed. The figure shows a modification to the first drawing, where the launch elevation is given as d_{vo} and the ending elevation is given as d_{vf}. If t_o is set equal to zero, any horizontal distance d_h can be written as a function of time:



$$d_h = v_o t\cos\theta \quad \text{or} \quad t = \frac{d_h}{v_o \cos\theta} \quad \text{and} \quad t_{of} = \frac{d_{h_{max}}}{v_o \cos\theta}$$

Any vertical distance d_v can likewise be written as

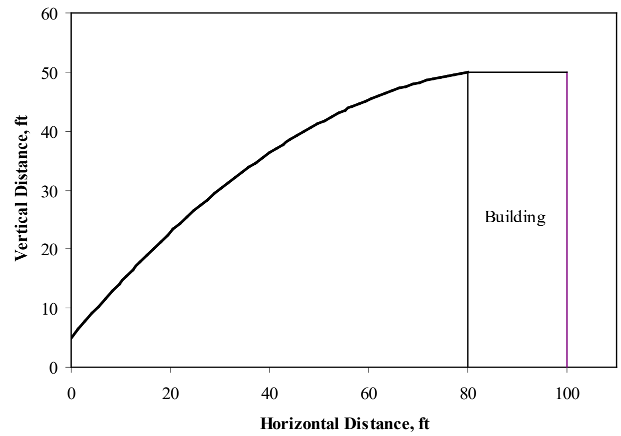
$$d_v = d_{vo} + v_o t\sin\theta + \frac{1}{2}gt^2$$

Trajectory Example 7. Major-league pitchers can throw a baseball about 90 mph. What is the time of flight of a ball thrown 530 ft at a release angle of 40º, measured from the pitcher to where it hits the ground? The pitcher releases the ball 5 ft above ground level. (s)

Using the horizontal distance equation solved for time, the time of flight is calculated.

$$t_{of} = \frac{d_{h_{max}}}{v_o \cos\theta} = \frac{530\text{ft}}{90\text{mph}\cos(40^\circ)}\left\{\frac{1\text{mi}}{5280\text{ft}}\right\}\left\{\frac{3600\text{s}}{1\text{hr}}\right\} = 5.24\text{s}$$

Trajectory Example 8. Hanna tosses a ball from the ground to the flat roof of a building 50 ft tall. She is 80 ft from the building and releases the ball at an angle of 45º relative to the ground and at a release elevation of 5 ft. What is the lowest acceptable initial velocity? (ft/s)

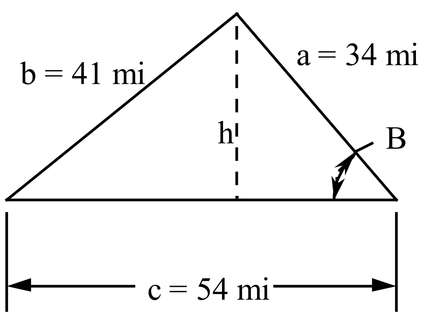


The lowest acceptable initial velocity will result in the ball just landing on the roof. That is, the maximum horizontal distance is 80 ft. The given information then is d_{vo} = 5 ft, d_{vf} = 50 ft, θ = 45º, d_{hmax} = 80 ft. Combining the d_h and d_v equations, we obtain

$$d_{vf} = d_{vo} + d_{h\,max} \tan\theta + \frac{g d_{h\,max}^2}{2 v_o^2 \cos^2\theta}$$

Substituting and solving for v_o yields v_o = 76.7 ft/s.

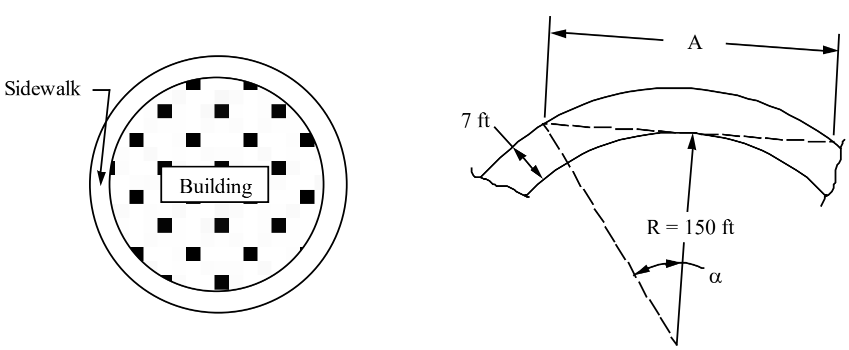
F. Problems Requiring Geometric Modeling



Geometry problems are listed separately on a Calculator Applications test, but many stated problems also involve geometric modeling. From the problem statement, you must render a described feature into a geometric model to solve for the unknown. You might have to model a garbage can as a frustum of a cone, a milk carton as a square right prism, the earth as a sphere, or a curved surface like the state of Nevada as a planar trapezoid! The rich variety of objects we use in our daily living provides an equally rich opportunity for geometric modeling. Essentially, knowledge of plane and solid geometry is needed, including calculation of edge length, periphery, areas and volumes. Trigonometry is also needed.

As the following example problems illustrate, the key consideration in working stated problems involving geometric modeling is to form the "reality" in the problem statement into a geometric form during or preceding problem translation. Once this is done, you can bring the appropriate formulas to bear on the model to arrive at a solution for the unknown. In some cases, the geometric model is obvious and accurate. Other times, it is not so apparent which form to assume.

Geometry Example 1. A cylindrical building 300 ft in diameter has a 7-ft sidewalk around it. If two persons do not leave the sidewalk, what is the maximum separation distance at which they can still see each other? (ft)



This problem illustrates some of the difficulties associated with geometric modeling. We don't know how the persons stand on the sidewalk other than that they want to be as far apart as possible. We suppose that they don't lean out into the street, and we know that a person has some non-zero width between 1 ft and 2 ft. These difficulties are mitigated somewhat by the desire to know the maximum distance between them. Looking down on the situation, we see a circular building of radius R equal to 150 ft, and a sidewalk of width w equal to 7 ft. The unscaled figure shows this as well as a more detailed plan of the modeling. We take the distance we desire A to be defined as shown since this is the maximum straight-line distance between points on the sidewalk. Half of A forms the leg of a right triangle, the other leg of which is R. The hypotenuse is R+w or 157 ft.

$$\left[\tfrac{1}{2}A\right]^2 + R^2 = (R+w)^2 \quad \text{or} \quad A = \sqrt{4[(R+w)^2 - R^2]} = \sqrt{4[157^2 - 150^2]} \text{ ft} = 92.7 \text{ ft}$$

Geometry Example 2. The distances between three towns is 34 mi, 54 mi and 41 mi. What area does this enclose? (sq.mi)

We model this as a scalene triangle of side dimensions, 34 mi, 41 mi and 54 mi. Trigonometry in general and trigonometry relating to scalene triangles are discussed in Chapter 5. This assumption of a scalene triangle is in fact an approximation since the "lines" of distance are actually curves on the surface of the earth which generate a spherical surface, not a flat one. Since spherical trigonometry is beyond the scope of the Calculator Applications Contest, we assume the curvature is small and therefore negligible. So, our model is a triangle, the side dimensions are known, and the unknown is the area. There is a formula called, "Heron's Formula", which gives the area of a scalene triangle in terms of the side dimensions. The student who knows this formula has a time advantage when working this type of problem, although the solution does not require such knowledge, as will be shown. For a scalene triangle of side dimensions a, b and c, Heron's Formula is

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = 0.5\,(a+b+c)$$

Substituting, we quickly get an answer.

$$s = 0.5\,(34 + 41 + 54) \text{ mi} = 64.5 \text{ mi}$$

$$A = \sqrt{64.5\,(64.5-34)(64.5-41)(64.5-54)} \text{ mi}^2 = 697 \text{ mi}^2$$

If Heron's Formula is not known to the contestant, the following approach will generate the answer. We reason, "To get the area, we need an altitude, like the one shown in the figure. To get this altitude, we need one of the angles such as Angle B. We can get this angle using the Law of Cosines if the three side dimensions are shown." The solution then becomes

$$b^2 = a^2 + c^2 - 2ac\cos B \qquad \text{[Law of Cosines]}$$

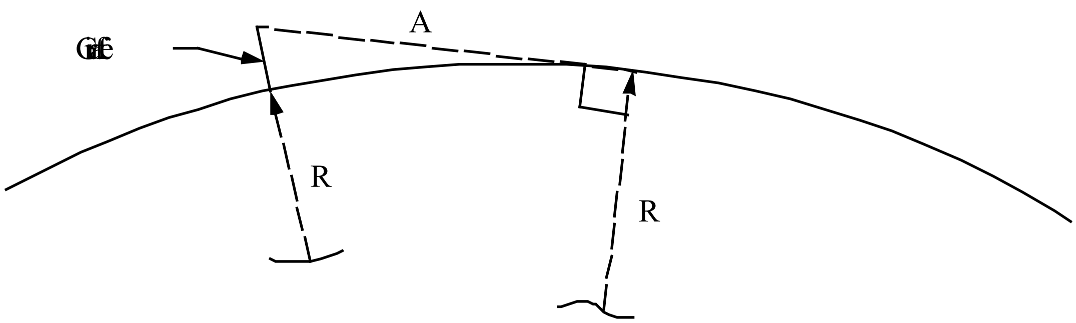
from which Angle B is solved.

$$B = \mathrm{Cos}^{-1}\left[\frac{b^2 - a^2 - c^2}{-2ac}\right] = \mathrm{Cos}^{-1}\left[\frac{41^2 - 34^2 - 54^2}{-2(34)(54)}\right] = \mathrm{Cos}^{-1}(0.651) = 49.37°$$

The altitude h shown in the figure is (a sinB), so the area is our answer A.

$$A = \tfrac{1}{2}\,c\,h = \tfrac{1}{2}\,c\,a\sin B = 0.5\,(54 \text{ mi})\,(34 \text{ mi})\,\sin(49.37°) = 697 \text{ mi}^2$$

Geometry Example 3. A full-grown male giraffe has an eye level 17 ft above the ground. How far away is the horizon to such an animal, assuming no refraction? (mi)



This style of geometric stated problem is common on Calculator Applications tests. The distance to the horizon is measured from the decks of ships, tall buildings, towers, airplanes, mountains, etc. The solution is common to all. The drawing shows the earth's surface of radius R equal to 3960 mi (Appendix A). You might think that the giraffe's 17-ft height is negligible relative to the radius of the earth, but this is not the case. The reason is that it is a first order term in the equation we are about to write. The right triangle is only partially drawn on the unscaled sketch. It has one leg of dimension R, one leg of dimension A (our answer), and the hypotenuse is R+17 ft. It helps to convert R to units of feet before solving the right triangle for A using the Pythagorean Theorem.

$$R \;=\; 3960\ \cancel{\text{mi}}\ \left\{\frac{5280\ \text{ft}}{1\ \cancel{\text{mi}}}\right\} \;=\; 20{,}908{,}800\ \text{ft}$$

$$A^2 + R^2 \;=\; (R + 17\ \text{ft})^2 \quad \text{or} \quad A \;=\; \sqrt{(R + 17\ \text{ft})^2 \;-\; R^2} \;=\; \sqrt{34R\text{ft} + 289\text{ft}^2}$$

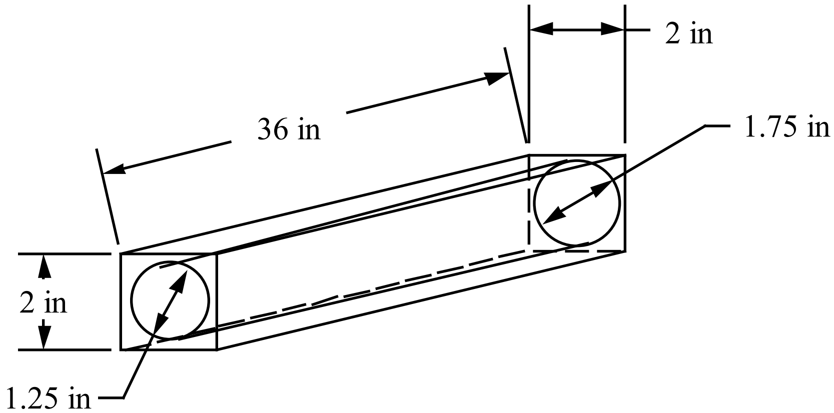
$$A \;=\; 26{,}663\ \cancel{\text{ft}}\ \left\{\frac{1\ \text{mi}}{5280\ \cancel{\text{ft}}}\right\} \;=\; 5.05\ \text{mi}$$

<u>Geometry Example 4.</u> The diameters of 12-gauge and 14-gauge wires are 0.08081 in and 0.06408 in, respectively. What length of 14-gauge wire weighs the same as 2,000 ft of 12-gauge wire? (ft)

We don't normally think of wire as a cylinder. In fact, wire is often coiled or spooled, but in the absence of information to help us make a more exact calculation, we assume a geometry associated with straight wire. This is certainly in error, because it is difficult to imagine a situation where 2,000 ft of wire is stretched out perfectly straight. This doesn't concern us here, because we figure that the error induced by coiling (or uncoiling) the wire has a negligible influence on its effective length. Assuming the wires are of identical material, the equality of weight means that their volumes must be equal. We solve for the length of the 14-gauge wire h_1 using this equivalency.

$$\frac{\pi}{4}\ D_1^2\ h_1 \;=\; \frac{\pi}{4}\ D_2^2\ h_2 \quad \text{or} \quad h_1 \;=\; h_2\left[\frac{D_2}{D_1}\right]^2 \;=\; 2{,}000\ \text{ft}\ \left[\frac{0.08081\ \cancel{\text{in}}}{0.06408\ \cancel{\text{in}}}\right]^2 \;=\; 3{,}180\ \text{ft}$$

<u>Geometry Example 5.</u> A woodworker places a 2 x 2 x 36 inch piece of wood on a lathe and turns it down to a conical table leg that is $1\frac{3}{4}$ inch diameter on one end and $1\frac{1}{4}$ inch diameter on the other. What percentage of the original wood was cut away? (%)



We have the geometry shown in the figure. The rectangular solid is the original volume V_o, and the table leg is a frustum. We must calculate the volume of both. The volume of the rectangular solid is (2 in)(2 in)(36 in) = 144 in^3. The volume of the frustum is given in terms of the base radii R_1 and R_2, and the height h.

$$V = \frac{\pi}{3}\, h\,(R_1^2 + R_2^2 + R_1R_2)$$

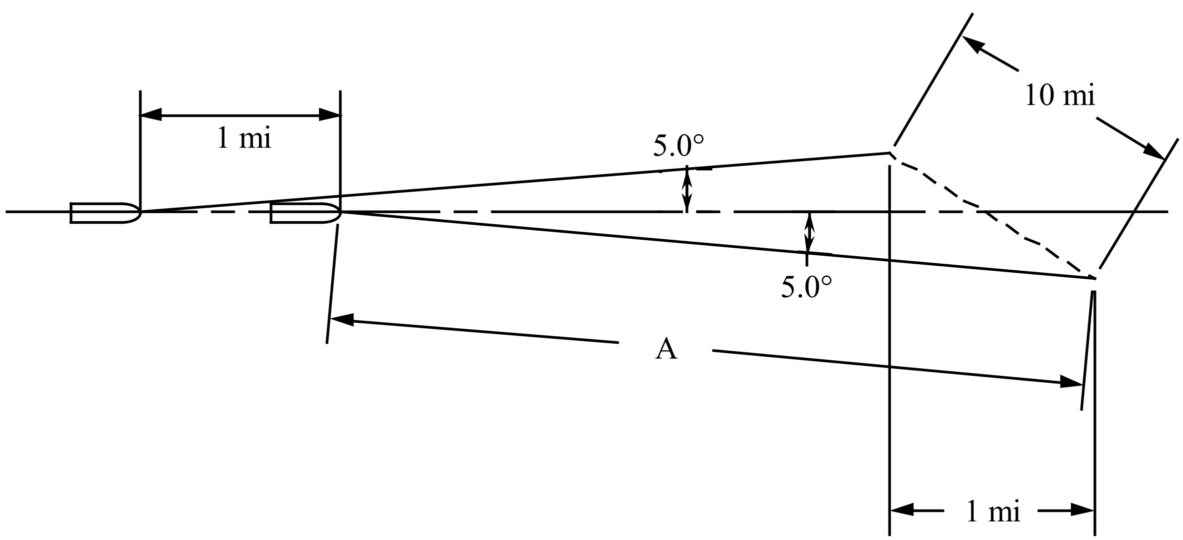
From this relation, the volume of the table-leg frustum V_{tl} is calculated.

$$V_{tl} = \frac{\pi}{3}\,(36\text{ in})\left[\left(\frac{1.75\text{ in}}{2}\right)^2 + \left(\frac{1.25\text{ in}}{2}\right)^2 + \left(\frac{1.75\text{ in}}{2}\right)\left(\frac{1.25\text{ in}}{2}\right)\right] = 64.206\text{ in}^3$$

The unknown is the percentage of material removed in the process. The volume cut away V_{lost} is the difference between the original volume V_o and the table-leg volume V_{tl}. The answer A is the fraction V_{lost}/V_o expressed as a percent.

$$A = 100\,\frac{V_{lost}}{V_o} = 100\,\frac{V_o - V_{tl}}{V_o} = 100\left[\frac{144 - 64.206}{144}\,\frac{\cancel{\text{in}^3}}{\cancel{\text{in}^3}}\right] = 55.4\%$$

Geometry Example 6. Two ships on maneuvers are cruising at the same speed in a straight line, with one ship one mile behind the other. At the same time, one turns starboard by 5° and the other port by 5°, and they continue on a straight course. How far does the front ship travel before they are 10 miles apart? (mi)



This is a complicated problem of plane geometry. The ships' horizontal (east-west in figure) distance h remains 1 mi as they travel at opposite angles away from each other. Their vertical (north-south in figure) distance v is 2Asin5°, where A, the answer, is the distance a ship travels after turning (The figure is not scaled). Their spacing of 10 mi forms the hypotenuse of a right triangle whose legs are h and v. From this, the value A may be obtained using the Pythagorean Theorem.

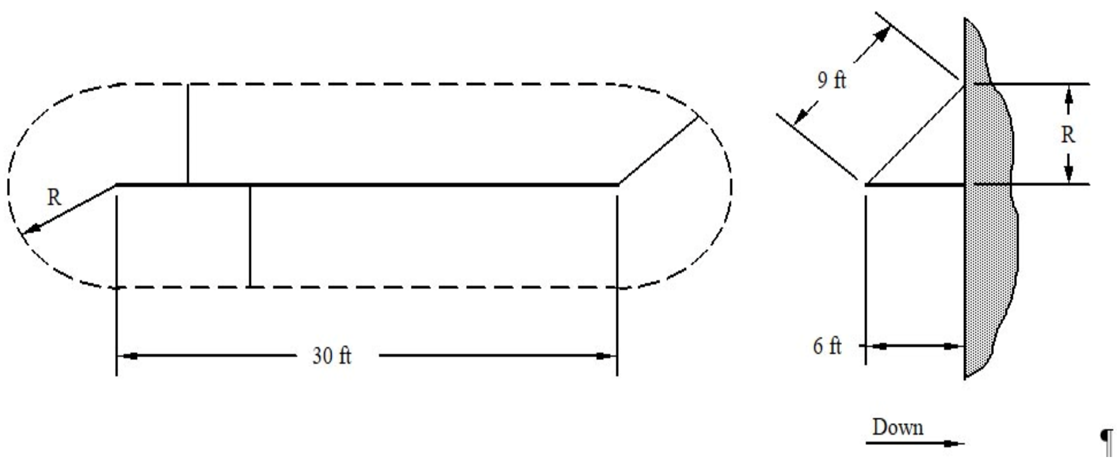
$$(10\text{ mi})^2 = h^2 + v^2 = (1\text{ mi})^2 + (2\text{ A}\sin 5°)^2 \quad\text{or}\quad 99\text{ mi}^2 = A^2\,(2\sin 5°)^2$$

$$A = \frac{\sqrt{99\ \text{mi}^2}}{2 \sin 5°} = 57.1\ \text{mi}$$

<u>Geometry Example 7.</u> One brand of dog run is a taut clothesline 30 ft long with the dog's 9 ft sliding leash attached to it. If the clothesline is 6 ft above the ground and does not sag, how much ground area can the dog roam? (sq.ft)

Neglecting such complications as tugging or stretching the clothesline, the height of the dog, and the fact that the leash is attached to the dog's neck, the roam space is the dotted periphery shown in the plan view below. The figure at the right shows the 9-ft leash and the 6-ft elevation. The direction of "down" is shown, so you may need to rotate the page clockwise 90° to help visualize the geometry of the situation. These dimensions form a right triangle with the remaining leg being the radius R of the dog run area. We need to know the radius, so we use the Pythagorean Theorem.

$$R = \sqrt{(9\ \text{ft})^2 - (6\ \text{ft})^2} = \sqrt{81 - 36}\ \ \text{ft} = 6.708\ \text{ft}$$



The area A is the area of a circle of this radius (or two semicircles of this radius) added to the rectangular area of side dimensions 30 ft and 2R.

$$A = \pi\,(6.708\ \text{ft})^2 + 2\,(6.708\ \text{ft})(30\ \text{ft}) = 544\ \text{ft}^2$$

G. Problems Involving Functions

Loosely speaking, a function is a relationship between two mathematical quantities. A simple example is $y = x^2$ for all x. This says, in effect, "You tell me what x is, and I'll tell you what y is." In this example, x is called the independent variable, since we get to choose whatever we want for its value, and y is called the dependent variable since its value depends on the value we selected for x.

Six specific function type problems are found on the Calculator Applications Contest. They fall into these categories: features on a graph, equation writing, compound interest (also exponential growth/decay), linear interpolation/extrapolation, percent problems and logarithmic solutions. As each category includes its own specific mathematical relationship between the independent and dependent variable, these categories will now be treated separately. In all cases, the approach to problem solving is to write the mathematical expression between x and y, called the "functional relationship" or "function", and to solve for the unknown.

i. Features on a Graph

These problems deal with points, lines, slopes, curves and enclosed areas on a Cartesian graph. The normal x-axis/y-axis convention is adopted. A point is usually referred to by the x and y value given in parentheses, "(x,y)". The "origin" of the coordinate system is the point (0,0). The equation for a line is

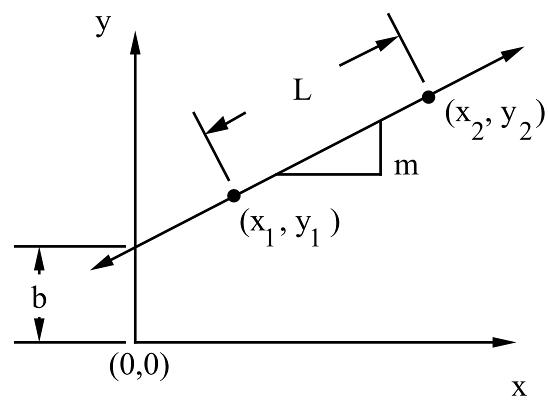
$$y = mx + b$$

where m is defined to be the slope of the line and b is called the y intercept (the y value when x is 0). The y intercept is the y value where the line crosses the y axis. On the figure, two points are noted, (x_1,y_1) and (x_2,y_2). The line on which these points lie is given with a slope m and y intercept b given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{and} \quad b = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \quad .$$

The length of the line segment L between the two points is given by the Pythagorean sum of the horizontal distance $(x_2 - x_1)$ and the vertical distance $(y_2 - y_1)$:

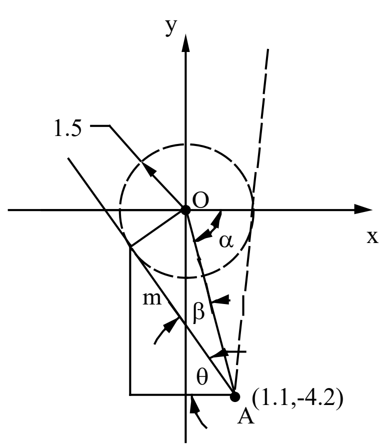
$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad .$$



Function problems often deal with curves, usually the class referred to as "conic sections": circles, ellipses, parabolas and hyperbolas. When this happens, the equation is always given as a functional relationship,
$y = -38x^2$ for a parabola, for example, or $2x^2 + 3y^2 = 45$ for an ellipse.

The only requirement for three-dimensional coordinate systems is to know the length of a line segment. This is necessary not only for function problems but also for geometric problems involving solid geometry. The coordinate system has three coordinates, (x, y, z). We think of x as being an east-west direction, y is the north-south direction and z is the up-down direction. Two points in space may be notated similarly to the two-dimensional case as (x_1,y_1,z_1) and (x_2,y_2,z_2). The length between the two points is given as the Pythagorean sum of the two horizontal distances $(x_2 - x_1)$ and $(y_2 - y_1)$, and the vertical distance $(z_2 - z_1)$:

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad .$$



Function Example 1. A straight line passes through the point x = 1.1, y = -4.2, and passes within 1.5 units from the origin at its closest approach. Find the slope of the line, assuming it to be negative.

A graph of the problem is shown. All points lying at a distance 1.5 units from the origin trace a circle of radius 1.5. The point given by (1.1,–4.2) is labeled "A" and the origin is labeled "O". There are actually two lines passing through Point A and within 1.5 units of the origin, but one of these (dotted) has a positive slope and is neglected. The length of Line Segment OA is L:

$$L = \sqrt{1.1^2 + (-4.2)^2} = 4.342 \quad .$$

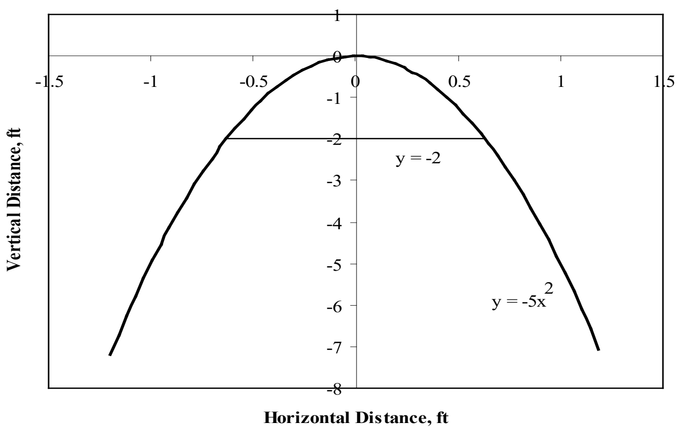
The measure of Angles α and β come from respective right triangles:

$$\alpha = \mathrm{Tan}^{-1}\left[\frac{4.2}{1.1}\right] = 75.32° \qquad \beta = \mathrm{Sin}^{-1}\left[\frac{1.5}{L}\right] = 20.21°$$

Since a line intersects parallel lines at the same angle, $\alpha = \theta + \beta$. Therefore, $\theta = \alpha - \beta = 75.32° - 20.21° = 55.11°$. The slope of the line is the vertical rise of a line divided by the horizontal run. From the figure, this "rise over run" is the tangent of Angle θ, and since the slope is negative, the answer becomes

$$\text{Slope} = -\tan(\theta) = -\tan(55.11°) = -1.43$$

Function Example 2. The parabola $y = -5x^2$ intersects the line $y = -2$ in two places. What is the distance between the two intersections?



Making a graph of each equation is a clear method to approach the problem. The fact that both equations are already solved for y allows us to obtain the solution rapidly using algebra. Substituting for y in the two equations yields

$$-2 = -5x^2$$

$$2/5 = x^2$$

$$x = \pm\sqrt{2/5} = \pm 0.632$$

The coordinates of the two intersections are (–0.632,–2) and (+0.632,–2). The distance between these two points L is the answer.

$$L = \sqrt{[0.632 - (-0.632)]^2 + [-2 - (-2)]^2} = \sqrt{[2(0.632)]^2 + 0^2} = 1.26 \quad .$$

ii. Equation Writing

The rapid way to a solution of function problems of this type is to write the functional relationship needed to determine the unknown. Sometimes the equation is given as part of the problem statement, and sometimes you need to write the equation based on information given in the problem statement. For these problems as well as others, it is particularly useful to ask the question, "What is the unknown?" or "What is the answer I seek?" This guides the formation of the equation needed to obtain an answer.

Function Example 3. Grains in zirconia of initial diameter D_o coarsen with time t to a new diameter D according to $D^3 - D_o^3 = kt$, where k equals $5.9\text{x}10^{-26}$ m^3/s. How long does it take to double the grain size for an initial value of 35 nanometers? (hours)

The unknown is time t. The equation is given explicitly, and we know that $D = 2D_o$ with $D_o = 35$ nm $= 35\text{x}10^{-9}$ m. Substituting, the answer is readily obtained.

$$D^3 - D_o^3 = (5.9\text{x}10^{-26}\ m^3/s)\ t$$

$$(70\text{x}10^{-9}\ m)^3 - (35\text{x}10^{-9}\ m)^3 = (5.9\text{x}10^{-26}\ m^3/s)\ t$$

$$t = \frac{3.001\text{x}10^{-22}\ \cancel{m^3}}{5.9\text{x}10^{-26}\ \cancel{m^3}/s} = 5{,}087\ \cancel{s}\left\{\frac{1\ hr}{3600\ \cancel{s}}\right\} = 1.41\ hr \quad .$$

Function Example 4. Momentum is the product of an object's mass times its velocity. A 15-gram bullet traveling at 500 m/s hits a 150-pound person wearing a bullet-proof vest. What is the backward velocity of the person if all the momentum is transferred? (m/s)

The unknown is the person's velocity which we may obtain from the momentum. The equation comes from a rigid translation of the first sentence; namely, momentum M equals the product of mass m and velocity r:

$$M = mr \quad .$$

The momentum of the bullet M_b is (15 g)(500 m/s) = 7500 gm/s. If all this is transferred to the person, then the person's momentum M_p may be expressed in terms of the person's velocity r_p:

$$M_p = M_b = (150\ \text{lb})(r_p) = 7500\ \text{gm/s}$$

Solving for r_p and doing unit conversions,

$$r_p = \frac{7500\ \cancel{\text{g}}\text{m/s}}{150\ \cancel{\text{lb}}}\left\{\frac{1\ \cancel{\text{lb}}}{453.592\ \cancel{\text{g}}}\right\} = 0.110\ \text{m/s} \quad .$$

iii. Compound Interest/Exponential Growth and Decay

When money is placed in a savings account at a bank, interest is paid into the account at some regular time interval. Suppose that the interest rate is "i", expressed often as a percent and paid over a time interval t. The amount of money initially placed in the bank is called the principal P_o. If the money is kept in the bank a total period of time nt where n is the number of time intervals, then the total money available at the end of this time P is given by the relation

$$P = P_o(1 + i)^n$$

Function Example 5. How much money must a person invest at 9% annual interest to have 1 million dollars in 40 years? ($)

In this problem, the principal P is unknown, i = 0.09, n = 40 and A = $1,000,000. Note that i must be in fractional, non-percent form when substituted into the interest equation.

$$P = P_o(1 + i)^n$$

$$\$1{,}000{,}000 = P_o(1 + 0.09)^{40} = 31.41\ P_o$$

$$P_o = \$31{,}837.58$$

Remember that for "$" problems, where the "$" appears in the answer blank, the answer must be written to the nearest penny independent of the number of significant digits involved.

Often, we refer to i as an annual compound interest rate paid q times per year (e.g., q = 4 for quarterly interest accrual, or q = 12 for monthly accrual). The number of time intervals n equals mq if m is the number of years, and the amount P is

$$P = P_o(1 + i/q)^{mq} \quad .$$

Function Example 6. A bank pays 8% on savings accounts, compounded quarterly. It lends money at an annual interest rate of 12%. What does the bank earn in the first year on a deposit of $5,000 assuming it is lent out for the same period and compounded monthly? ($)

The bank makes money on the $5000 principal. The final amount P_m is given below for monthly compounding (i.e., q = 12) and a one-year duration (n = 1).

$$P_m = P_o(1 + i/q)^{nq} = \$5000(1 + 0.12/12)^{12} = \$5634.12515$$

The bank must pay out 8% interest to the depositor, compounded quarterly (q = 4). At the end of one year, the account must have P dollars in it.

$$P = P_o(1 + i/q)^{nq} = \$5000(1 + 0.08/4)^4 = \$5412.16080$$

The bank's earnings is the difference in these two values.

$$\text{Earnings} = \$5634.12515 - \$5412.16080 = \$221.96$$

These relations may be used in the solution of problems of inflation, where the relative cost of items is at issue. A matinee movie once cost \$0.12, whereas now the same movie might cost \$12.00. We wonder what it will cost in the year 2075? This question can be answered by starting with the first equation for P and assuming an annual inflation rate i, with the initial cost equal to P_o and the cost after n years equal to P. When i is negative, the cost declines with time and we refer to the situation as deflation rather than inflation.

Actually, anything that multiplies itself by increasing by a percentage over some time interval can be modeled as a compound interest problem. Examples might be the interest accruing monthly on an unpaid credit card account, a person's salary given periodic raises, the number of bacteria growing in a culture, problems dealing with inflation or deflation, or the population growth or decline of a city, state or country.

Function Example 7: In 1959, gasoline was 20.9¢/gal. In 1992, it was \$1.21/gal. What is the annual inflation rate for gasoline? (%)

The number of years n is 1992 – 1959 = 33. The new value A is \$1.21/gal, and the initial value P_o is \$0.209/gal. The solution of i is shown.

$$P = P_o(1 + i)^n$$

$$\$1.21/\text{gal} = \$0.209/\text{gal}\ (1 + i)^{33}$$

$$(1 + i) = (1.21/0.209)^{1/33} = 1.05465$$

$$i = 0.05465 = 5.47\%$$

Function Example 8. A new employee started at \$25,000 annual income. If a raise averaging 3 percent per year is given, what will the annual salary be at retirement after 40 years? (\$)

The initial value is P_o = \$25,000, the annual interest rate i equals 0.03, and the number of time intervals for incrementing salary is n = 40. The solution for P is

$$P = P_o(1 + i)^n$$

$$P = \$25{,}000(1 + 0.03)^{40}$$

$$P = \$81{,}550.94$$

A special class of compound interest problems are exponential growth or decay. In a time period t_o, the amount of something changes from an initial value P_o to a new value P. From this, a general growth or decay relation may be written relative to an arbitrary starting value P_{o1} which grows or decays to a corresponding final value P_1 after a time period t:

$$\frac{P}{P_o} = \left[\frac{P_1}{P_{o1}}\right]^{t/t_o}$$

Function Example 9. A certain bacteria doubles in number every 5 hr. If there were 367 on a culture dish at 7:23 PM, how long will it take for the bacteria to multiply to 1 million? (hr)

Here the initial amount P_{o1} increases to $P_1 = 2P_{o1}$ in a time interval $t_o = 5$ hr. From this, we may write a general formulation for exponential growth:

$$\frac{P}{P_o} = \left[\frac{P_1}{P_{o1}}\right]^{t/t_o} \quad \rightarrow \quad P = P_o\left[\frac{2P_{o1}}{P_{o1}}\right]^{t/5\,hr} = P_o\left[2^{t/5\,hr}\right]$$

From other given information, we know that we start with P = 367 bacteria and after time t we have $A = 1x10^6$ bacteria. These values may be substituted into the equation and t may be solved. We must use a property of logarithms (base e or 10). It is

$$\text{Log}\,(K^m) = m\,\text{Log}(K) \quad .$$

The above equation may now be solved.

$$1x10^6 = 367\,[2^{t/5hr}]$$

$$\frac{1x10^6}{367} = 2^{t/5hr}$$

$$\text{Log}\left[\frac{1x10^6}{367}\right] = \text{Log}[2^{t/5hr}] = \left(\frac{t}{5hr}\right)\text{Log}(2)$$

$$t = \frac{\text{Log}\left[\frac{1x10^6}{367}\right]}{\text{Log}(2)}\,(5\text{ hr}) = 57.1\text{ hr}$$

A special case for exponential growth or decay is based on the natural logarithm. Any growth/decay equation can be converted to this form using the relationship

$$P = P_o e^{t/\tau} \;\; where\; \tau = \frac{t_o}{ln(P_1/P_{o1})}$$

For the previous problem,

$$\tau = \frac{5\,hr}{ln(2)} = 7.21\,hr$$

The solution would then be written in this standard exponential form as

$$P = P_o e^{t/7.21\,hr} \;\; or \;\; 10^6 = 367e^{t/7.21\,hr}$$

$$ln\left(e^{t/7.21\,hr}\right) = \frac{t}{7.21\,hr} = ln\left[\frac{10^6}{367}\right] = 7.966$$

$$t = 57.1\text{ hr}$$

iv. Linear Interpolation and Extrapolation

As already described, function problems deal with functional relationships between two variables such as x and y. The relationship is normally written as an equation. Often, the equation is unknown, but a pair of values (x_1,y_1) and (x_2,y_2) are known. Linear *interpolation* is a method for obtaining a value of y for a value of x such that $x_1 \le x \le x_2$. Linear *extrapolation* is used to find a y when x does not lie in the aforementioned domain.

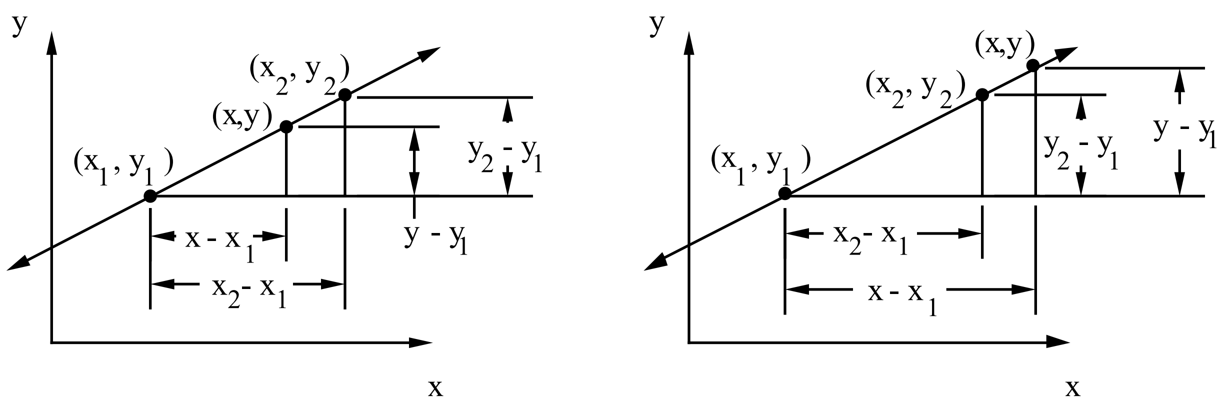
The method entails plotting the two points as (x,y) values on a coordinate system, without regard to the fact that x and y are generally not distances. The new value of y is interpolated or extrapolated by associating it with the new x value which lies on the line. The figure shows this relationship.

Many calculators have the built-in capabilty to perform interpolation and extrapolation automatically, and contestants are encouraged to learn how to use this feature. The fundamental equations are derived using either the interpolation figure or the extrapolation figure. Two similar right triangles may be discerned. Writing the ratios of their legs, one obtains

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$$

from which we get the following solution for y:

$$y = y_1 + \frac{(y_2 - y_1)(x - x_1)}{x_2 - x_1} \quad .$$



Interpolation   Extrapolation

Function Example 10. On average, boys weigh 8 pounds at birth and 140 pounds at age 18. What is the average weight of a 14 year old? (lb)

The parameters being related in this problem are the weight of a boy and his age. Letting x be the age and y be the weight, we have two (x,y) pairs of (0 yr, 8 lb) and (18 yr, 140 lb). It doesn't matter which pair is (x_1, y_1) and which is (x_2, y_2). The value of x is 14 yr, so what is y?

$$y = y_1 + \frac{(y_2 - y_1)(x - x_1)}{x_2 - x_1} = 8 \text{ lb} + \frac{(140 - 8)\text{lb}(14 - 0)\text{ yr}}{(18 - 0)\text{ yr}}$$

$$y = 8 \text{ lb} + 102.7 \text{ lb} = 111 \text{ lb}$$

v. Percent Problems

When two numbers are compared, one effective method involves taking a percentage. Care must be taken when doing this if a unique answer is desired. Consider this problem: "What is the percent error in approximating 30 cm to be 12 in?" First, we must obtain a common length measure by converting one number. Arbitrarily choosing the 12 in for conversion, we find that

$$12\ \not{\text{in}} \left\{\frac{2.54\ \text{cm}}{1\ \not{\text{in}}}\right\} = 30.48\ \text{cm} \ .$$

There are four possible answers to our problem. The first is to subtract the two numbers to obtain 0.48 cm and then to divide by 30.48 cm to obtain 0.0157. Converting to a percentage, the answer becomes 1.57%. We could just as well divide the difference by 30 cm, which yields a percentage of 1.60%, a second answer. Two more answers are the negatives of these first two answers, as there is no reason why the difference could not be taken as 30 cm – 30.48 cm.

Since we have four answers, a conventional definition of terms is required to specify exactly which answer is appropriate for the Calculator Applications Contest. In doing so, four types of percentage problems have been developed, and the language is specific within the contest. We use standard conventions in engineering and science to form the relationships. These problems are percent increase, percent error, percent change, and percent decrease. It is easy to distinguish these problems. When the unknown is requested in a problem statement, the type of percent will be explicitly stated. Further, a "%" will appear in the answer blank.

The first type of percentage problem is the *percent increase* problem. "Percent increase" implies that a smaller number has gotten bigger, so it makes sense that the smaller number would be the basis for comparison. It is the intent of the problem style that the percent increase answer would always be positive, as in real-life situations it is not normally sensible to talk about a negative increase in a number. Rather, we would use the (positive) percent decrease fome.

$$\text{Percent Increase} = 100\left[\frac{\text{Larger Number} - \text{Smaller Number}}{\text{Smaller Number}}\right] = 100\left[\frac{\text{Larger Number}}{\text{Smaller Number}} - 1\right] \ .$$

While the first part of the equation demonstrates the concept of a percent increase better than the last term, we strongly recommend use of the second formulation in problem solving. First, it requires fewer key strokes, an important speed consideration. Second, when working significant-digit stated problems, errors in the significant-digit part of the answer sometimes arise when numbers are repeated in the calculation. This is discussed in greater detail in the section on significant-digit stated problems. Clearly, the number which ultimately appears on the calculator display is the same independent of which equation is used.

In *percent error* problems, one number will be exact and the other will be an approximation. The important change is relative to the exact number so it is the basis for comparison. If the exact and approximate numbers are represented by variables E and A, respectively, the percent error is given by the formula

$$\text{Percent Error} = 100\left[\frac{A - E}{E}\right] = 100\left[\frac{A}{E} - 1\right] \ .$$

With *percent change* problems, there is a clear "starting number" or "important number" defined by the context of the problem. This becomes the basis (denominator) for the percent equation. Consider the problem, "What is the percent change in the mass of a shrimp that is first 'small' and later grows to 'jumbo' size? There are 55 small shrimp in a pound but only 21 jumbo shrimp per pound. The context clearly has shrimp starting as small and later becoming jumbo. So the basis is the small shrimp mass (1/55 lb for each small shrimp). The equation and solution are

$$Percent\ Change = 100\left[\frac{Final - Starting}{Starting}\right] = 100\left[\frac{Final}{Starting} - 1\right]$$

$$Percent\ Change = 100\left[\frac{1/21}{1/55} - 1\right] = 162\%$$

The last type of percentage problem is a *percent decrease* problem. Similar to percent increase problems, the implication here is that a number is diminishing in size. It becomes sensible in this case to consider the larger number as the basis for comparison, which in the above equations would always yield a negative answer. As the negative sign no longer carries any meaning, it is dropped, making the percent decrease always positive.

$$\text{Percent Decrease} = 100\left[1 - \frac{\text{Smaller Number}}{\text{Larger Number}}\right] \quad .$$

In closing this section on percentage problems, a summary table has been prepared to help condense the numerous relations described above into a single form which will hopefully assist the contestant. First, realize that two numbers are involved in percent problems. Let the number forming the basis for comparison be B, and the number changing relative to this basis be N. Then, when a percentage problem is given, replace N and B in the equation shown below with the appropriate term in the table. The remaining number in the problem statement is N by default .

$$\text{Percent Error} = 100\left[\frac{\text{N}}{\text{B}} - 1\right]$$

Problem Type	Replace B With
Percent Error	Exact Number
Percent Change	Starting or Important Number
Percent Increase	Smaller Number (Answer always positive)
Percent Decrease	Larger Number (Answer always positive)

Function Example 11. The shortest river in Texas is the Comal River, 2.5 mi long. A foreign visitor considered it to be 4 km long. What is the percent change between these values? (%)

First, we convert to a common length dimension such as miles.

$$4\text{ km}\left\{\frac{1\text{ mi}}{1.6093\text{ km}}\right\} = 2.485\text{ mi}$$

The solution then becomes

$$Percent\ Change = 100\left[\frac{Final}{Starting} - 1\right] = 100\left[\frac{2.485}{2.5} - 1\right] = -0.600\%$$

Function Example 12. The surface tension of water at 10°C and 30°C is 74.22 dyne/cm and 71.18 dyne/cm, respectively. What is the percent error in the interpolated value at 20°C, given the actual value of 72.75 dyne/cm? (%)

The actual, or "exact", value is given, but we must first interpolate the approximate value. Using the appropriate equation from the earlier section on interpolation and extrapolation, we obtain

$$y = y_1 + \frac{(y_2 - y_1)(x - x_1)}{x_2 - x_1} = 74.22\text{ dyne/cm} + \frac{(71.18 - 74.22)\text{dyne/cm}(20 - 10)°\text{C}}{(30 - 10)°\text{C}} = 72.70\text{ dyne/cm.}$$

Substitution into the percent error equation yields the answer.

$$\text{Percent Error} = 100\left[\frac{A}{E} - 1\right] = 100\left[\frac{72.70\text{ dyne/cm}}{72.75\text{ dyne/cm}} - 1\right] = -0.0687\%$$

Function Example 13. What is the percent increase in the volume of a right circular cone if the height is increased by 5% and the diameter of the base is increased by 6%? (%)

We must first calculate the volumes of the initial and enlarged cones. The initial volume V_o of a cone of base diameter D and height h is

$$V_o = \frac{\pi}{12}\,D^2h$$

If the height is increased by 5% and the base diameter increased by 6%, the new volume V_n is

$$V_n = \frac{\pi}{12}(1.06D)^2(1.05h) = 1.1798\,\frac{\pi}{12}\,D^2h = 1.1798\,V_o \quad .$$

Substituting into the percent increase formula, the answer is obtained.

$$\text{Percent Increase} = 100\left[\frac{\text{Larger Number}}{\text{Smaller Number}} - 1\right] = 100\left[\frac{1.1798\,V_o}{V_o} - 1\right] = 18.0\%$$

Function Example 14. A wire under tension stretches 10%. What was the percent decrease in radius, assuming constant volume? (%)

Assume that the wire has a radius R and length L. For a cylindrical shape,

$$V = \pi R^2L$$

If the wire is stretched 10%, the length increases to a new value $L_s = 1.1L$, and the radius decreases to R_s. R_s may be obtained from the equality of volume.

$$V = \pi R^2L = \pi R_s^2L_s = \pi R_s^2(1.1\,L)$$

$$\pi R^2L = \pi R_s^2(1.1\,L)$$

$$R^2 = 1.1\,R_s^2$$

$$R_s = 0.9535\,R$$

The answer then comes from use of the percent decrease formula.

$$\text{Percent Decrease} = 100\left[1 - \frac{\text{Smaller Number}}{\text{Larger Number}}\right] = 100\left[1 - \frac{0.9535\,R}{R}\right] = 4.65\%$$

vi. Logarithmic Solutions

Calculators have problems with extremely small or extremely large numbers. Usually, the calculator substitutes "0" for small numbers, and the largest number it knows for large numbers, often $9.99\text{x}10^{99}$. Some calculators print a message, like "Out of Range" or "Underflow". You can see what your calculator does by trying to obtain the answer to 577^{3094} or 1.45^{-7692}. The purpose of logarithmic solutions is to extend the capacity of your calculator beyond its intrinsic range. To do this, knowledge of two properties of logarithms is necessary. These properties are

$$Log_{10}(K^m) = m\,Log_{10}(K) \quad \text{and} \quad 10^{log_{10}(A)} = A \;.$$

We use these relationships to split a large number into two workable numbers, the digits part of the number and the power of ten. To evaluate a number of the form K^m using the logarithmic solution, we write an equation where the answer A is given.

$$A = K^m$$

Next, we take the base 10 logarithm of both sides. With use of the property of logarithms, this yields:

$$Log_{10}(A) = Log_{10}(K^m) = m\,Log_{10}(K) = M + C,$$

where the last term has been split into two parts: a number M, called the "mantissa", which must lie between zero and positive one, and an *integer* remainder C, called the "characteristic". When A is greater than 1, M is the fractional part and C is the integer part of $m\,Log_{10}(K)$. When A is between zero and one, C must be one less than the integer part of $m\,Log_{10}(K)$. The motivation for this will become apparent in the example problems below. We now raise 10 to a power equal to the left and right terms of the above equation. Another way to say this is to take the "anti-logarithm" of both sides of the equation.

$$10^{log_{10}(A)} = 10^{(M + C)}$$

$$A = 10^M x 10^C$$

The only remaining task is to evaluate 10^M on the calculator, leaving 10^C as the power-of-tens part of a scientific notation answer. 10^M will always be positive and between 1 and 10. The best way to demonstrate the utility of the logarithmic solution is through several illustrative examples.

<u>Function Example 15.</u> What is 5050^{505}?

Following the steps above, an answer is obtained.

$$A = 5050^{505}$$

$$Log_{10}(A) = Log_{10}(5050^{505}) = 505\,Log_{10}(5050) = 1870.16215 = 0.16215 + 1870$$

$$10^{log_{10}(A)} = 10^{(0.16215 + 1870)}$$

$$A = 10^{(0.16215 + 1870)} = (10^{0.16215})10^{1870}$$

$$A = 1.45x10^{1870}$$

<u>Function Example 16.</u> What is 7080^{-655}?

The solution is identical to the previous example, excepting that care must be taken in choosing the correct value of the mantissa M, the fractional part of the logarithm of A.

$$A = 7080^{-655}$$

$$Log_{10}(A) = Log_{10}(7080^{-655}) = -655\,Log_{10}(7080) = -2521.772 = 0.228 + (-2522)$$

Note that M must be between 0 and 1, so –0.772 is not the correct value for M. We must add one to –0.772 to obtain 0.228 and maintain equality by subtracting one from –2521 to obtain C = –2522.

$$10^{\log_{10}(A)} = 10^{[0.228 + (-2522)]}$$

$$A = 10^{[0.228 + (-2522)]} = (10^{0.228})10^{-2522}$$

$$A = 1.69 \times 10^{-2522}$$

H. Problems Involving Transcendental Functions (Solver Problems)

There are a large number of engineering equations that involve one variable and which do not have an algebraic solution. Consider x+sinx=2 where x is in radians. There are advanced methods for solving this using series expansions of the sin function, but these methods lie outside the purview of the Calculator Applications Contest. Equations like this can be solved using the solver (or equivalent) feature on a calculator.

The solver approach is to guess a value for x and see how close we come to making the equation work. For the function x+sinx=2, we might guess x=1.5. Then x+sinx=2.50. If we guess x=2, we get x+sinx=2.91. Based on these two values, we surmise that x might be less than 1.5, so we guess x=1 and get x+sinx=1.84. So, it looks like x lies between 1 and 1.5. By guessing1.25, etc., to continue to refine the value of x, we could with patience arrive at x=1.11, for which x+sinx =2.000. This educated-guess approach to solving an equation is called *iteration*.

Engineering calculators have a built-in program (the solver) that works by doing the educated guessing for us very quickly. The method varies depending on the calculator, but the general steps to using a solver are (a) enter the equation into the calculator, (b) setting a first guess for the variable we want to solve, and (c) instructing the calculator to solve the equation for the desired variable. To learn specifics on the use of the solver for your calculator, you must refer to the operator's manual or ask your coach or another person who knows how to use the solver feature.

In general, one must be careful with solvers when more than one solution exists. For example, suppose we entered a quadratic equation like $3x^2$-7x+18=0. This is not a transcendental equation, but the solver can be helpful for solving quadratics if the solution to the quadratic equation is not a built-in feature of the calculator. By setting the initial x value to 1, we get an iterated solution of x=1.55, but we know that there's a second root out there somewhere, so how do we find it? One way is to guess a new initial value of x and see if the second solution is found. Usually, very large positive or negative guesses work. By using an initial value of x=-100, the solver solves for x=-3.88, the other root. Some calculators have solvers that will solve for all the roots of an equation at once (very nice). Others will only find a nearby root based on the initial value of the variable.

Multiple answers are possible with transcendental functions too. Consider the function we started with modified by squaring one x: x^2+sinx=2. By starting with an initial value of x=1.5, we obtain an iterated solution of x=1.06. However, there is also a solution at x=-1.73, but that is not so apparent from just looking at the function. This creates an issue for test *writers*, but not test *takers*. Problems created for the contest must have single answers. There are two ways to handle this. First, x can be bounded in the problem statement. That is, the problem could read, "For what value of x between 1 and 10 does x^2+sinx=2?", or "For what negative value of x does x^2+sinx=2?" These problem statements should clue the contestant to set the initial value of x in the range. For example, for the first form of the question, an initial value of x=5 might be appropriate. An alternative approach is to make sure there is only one solution. For the x^2+sinx=2 problem, we can accomplish this by removing the square from x as was done at the start of this section. A good practical procedure to insure a single-valued solution is to plot the homogeneous form of the function (i.e., x+sinx-2=0) over a large range of x, knowing that solutions are where the function crosses the x axis. This, coupled with elementary analysis of the functions themselves extended beyond the plot range, will help convince a problem creator that in fact a single solution exists. Test *takers* should appreciate that a solver problem on the contest will have only one solution, but they should also appreciate that if they use the solver as a convenience on a non-solver problem on the test, there may be multiple answers.

Solver Example 1. Solve for x if $35x^{5.3} = \pi x^{2.1} + 75$.

By entering the equation and setting the initial value of x to 0, the solver produces 1.17.

Solver Example 2. For what negative, non-zero value of z (rad) does $20\sin(z) = -3z^2$?

First, the calculator is set to the radian mode for angles. Then, by entering the equation and guessing an initial value of x of -5, the solver produces -2.26.

Solver Example 3. Solve for t (deg) if $4\sin(18t)+30\cos(10t) = 2t-30$.

Making sure the calculator is set for degree measure, the equation is entered and, guessing an initial value of x = 0, the solver produces 10.6.

Currently, Problem #48 on UIL tests is always a solver problem. It will be "guaranteed" to have only one unique answer. However, there are potentially other problems on the test whose solutions are either accelerated by using the calculator solver feature or which effectively require use of the solver to obtain a solution. Contestants should be discerning in these cases to assure that the answer is sensible given the problem context, because multiple solutions may exist.

I. Problems Requiring Scaling Principles

The scaling approach to problem solving is very powerful because it allows us to obtain accurate answers even though we don't have enough information to completely solve for everything. The trick to scaling problems is first to realize that something in the problem statement is proportional to something else in the problem statement. We used this approach in the rotational motion part of the Rate Section to obtain $d = \mathbf{R}\theta$. In general, we have some value y that is directly proportional to x. We can always write this proportionality as an equation by inserting a constant of proportionality k to obtain y = kx. If in the problem statement we are given just one pair of values of x and y, here called x_1 and y_1, then we can produce a general equation relating x and y. We use x_1 and y_1 to solve for k, and then substitute this value back into the original equation to obtain the final form.

$$k = \frac{y_1}{x_1} \qquad \text{and} \qquad y = \left[\frac{y_1}{x_1}\right] x \qquad \text{or} \qquad \left[\frac{y}{x}\right] = \left[\frac{y_1}{x_1}\right]$$

The last equation forms a pair of ratios. The terms, "ratio of y to x" and "ratio between y and x" are used in this contest to apply to the result of the division y/x.

Scaling Example 1. The constant thickness rings on a tree may be used to determine a tree's age. If a 25 year old redwood is 15 inches in circumference, what is the age of a redwood 25 inches in circumference? (yr)

This problem has sufficient information to work without using scaling principles if we assume that the tree trunk is cylindrical in shape. This is not a bad assumption since the problem statement refers to the tree's circumference. There is still an advantage to working the problem using scaling, because as we will demonstrate, the solution may be obtained much more rapidly using scaling. First, we work the problem the "hard" way, without using scaling. The circumference c equals $2\pi R$ where R is the radius of the tree. The thickness of a ring h is the tree's radius divided by the age a. From these relations and the given information, we can solve for h.

$$h = \frac{R}{a} = \frac{c/2\pi}{a} = \frac{c}{2\pi a} = \frac{15 \text{ in}}{2\pi(25 \not{\text{yr}})}\left\{\frac{1 \not{\text{yr}}}{1 \text{ ring}}\right\} = 0.0955\,\frac{\text{in}}{\text{ring}}$$

The radius of the older redwood is obtained knowing the circumference is 25 in. The new radius is 25 in/2π or 3.979 in. Solving the above equation for the tree's age, we can substitute and solve.

$$a = \frac{R}{h} = \frac{3.979 \text{ in}}{0.0955 \text{ in/ring}} = \frac{3.979 \cancel{\text{in}} \cancel{\text{ring}}}{0.0955 \cancel{\text{in}}} \left\{\frac{1 \text{ yr}}{1 \cancel{\text{ring}}}\right\} = 41.7 \text{ yr}$$

Now, we shall demonstrate the solution using scaling principles. We note that the radius R is proportional to the circumference c, and also that the age of the tree a is proportional to the tree's radius. If $R \propto c$ (the symbol "$\propto$" means "is proportional to") and $a \propto R$, then $a \propto c$. Writing this in the general form of a scaling equation and substituting $a_1 = 25$ yr and $c_1 = 15$ in, we quickly obtain the answer, knowing that the new circumference c is 25 in.

$$\left[\frac{a}{c}\right] = \left[\frac{a_1}{c_1}\right] = \left[\frac{25 \text{ yr}}{15 \text{ in}}\right] \quad \text{or} \quad a = \left[\frac{25 \text{ yr}}{15 \text{ in}}\right] c = \left[\frac{25 \text{ yr}}{15 \cancel{\text{in}}}\right] 25 \cancel{\text{in}} = 41.7 \text{ yr}$$

Usually it is not possible to work the problem except by scaling, because the geometry is not given. Also, the proportionality does not have to be linear; that is, y could be proportional to x^n, where n could be any positive or negative number. When $n = -1$, we say that y is "inversely proportional" to x. Further, if $y \propto x$ and $y \propto z$, then we can also say that $y \propto (xz)$. The next example problem shows how this multiple-variable scaling works.

Scaling Example 2. Race drivers "lean into a curve" due to an imposed force proportional to the square of the driver's speed and inversely proportional to the curve radius. If the maximum speed for a 60-ft radius curve is 50 mph, how fast can a driver make a 95-ft radius curve? (mph)

The scaling equation relates force F to speed v and curve radius R.

$$F = k \frac{v^2}{R} \quad ,$$

where k is the unknown constant of proportionality. Since we are dealing with maximum speed values, this implies that the car and/or driver have a maximum force rating which may not be exceeded. For our purposes, F is constant and may be included in the constant of proportionality as the equation is solved for the unknown variable v.

$$v = \sqrt{\frac{F R}{k}} = \sqrt{\frac{R}{k'}} = k'' \sqrt{R}$$

Note how we can always replace the product of constants and any operation on constants with a new constant. Here k/F became k', and $1/\sqrt{k'}$ became k". Writing this as a scaling equation with v as "y" and $\sqrt{R}$ as "x", we can substitute the values from the problem statement and solve.

$$\frac{v}{\sqrt{R}} = \frac{50 \text{ mph}}{\sqrt{60 \text{ ft}}} \quad \text{or} \quad v = \left[\frac{50 \text{ mph}}{\sqrt{60 \cancel{\text{ft}}}}\right] \sqrt{95 \cancel{\text{ft}}} = 62.9 \text{ mph}$$

Some of the most interesting scaling problems deal with three-dimensional geometric figures that have the same shape but different size. When this occurs, we say that the figures are geometrically "similar". Examples of geometrically similar geometries are a child and a grown-up, a model of the Eifel Tower and the real thing, or perhaps a little fish and a big fish. Scaling principles allow us to work with length, area and volume interchangeably on similar figures, even though we don't know and can't describe mathematically the shape of the object! We will now present three scaling principles for geometrically similar figures followed by their application to a cylindrical shape and some example problems.

Principle 1. Length If x and y are length dimensions on geometrically similar figures, then y is proportional to x, then y = kx and

$$\frac{y_2}{y_1} = \frac{x_2}{x_1}$$

Principle 2. Area If x is a length and A is an area on geometrically similar figures, then $A = k'x^2$ and

$$\frac{A_2}{A_1} = \left[\frac{x_2}{x_1}\right]^2$$

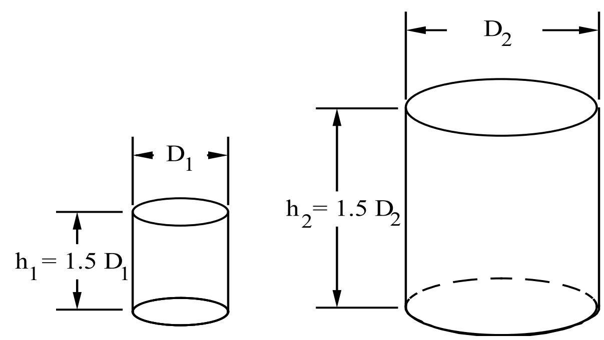
Principle 3. Volume (a) If x is a length and V is a volume on geometrically similar figures, then $V = k''x^3$ and

$$\frac{V_2}{V_1} = \left[\frac{x_2}{x_1}\right]^3$$

(b) The mass m of an object (or its weight) is the product of its density ρ and it volume V. From this, a similar relation between mass and any length dimension x may be written.

$$\frac{V_2}{V_1} = \frac{m_2/\cancel{\rho}}{m_1/\cancel{\rho}} = \frac{m_2}{m_1} = \left[\frac{x_2}{x_1}\right]^3 .$$

To demonstrate these principles, we consider the two cylinders shown. They are similar because they have the same shape; that is, they are both right circular cylinders with a height equal to 1.5 times the diameter. Scaling Principle 1 states that the ratio of any two lengths on the smaller figure equals the same ratio on the larger one. We've used one of these to pin down the shape; namely, that h = 1.5D. It doesn't matter how complex the length dimension is. The principle is valid for any length: height, diameter, circumference, the cylinder's diagonal, one fifth of the height added to 2 times the perimeter of an equilateral triangle inscribed on the cylinder's base, etc.



Scaling Principle 2 indicates that any area A equals any length dimension x squared. We know that the base area A_s for example equals $\pi D^2/4$. From this we can obtain the second scaling principle.

$$\frac{A_{s2}}{A_{s1}} = \left[\frac{\frac{\pi}{4}D_2^2}{\frac{\pi}{4}D_1^2}\right] = \left[\frac{D_2}{D_1}\right]^2 .$$

The length dimension doesn't have to be D. It can be any length. The concept may be shown by writing the relationship using the height by substitution of D = h/1.5.

$$\frac{A_{s2}}{A_{s1}} = \left[\frac{D_2}{D_1}\right]^2 = \left[\frac{h_2/\cancel{1.5}}{h_1/\cancel{1.5}}\right]^2 = \left[\frac{h_2}{h_1}\right]^2 .$$

Since we can relate any length to D by Scaling Principle 1, we can in similar fashion relate A_s to any length squared. The area can be any area we define. You might think that it doesn't work for all areas because, for example, the total area is $A_T = \pi D^2/2 + \pi Dh$, which does not fit our proportionality. The trick here is simply to substitute h = 1.5D into the equation and simplify. From this, Scaling Principle 2 is validated.

$$A_T = \frac{\pi D^2}{2} + \pi Dh = \frac{\pi D^2}{2} + \pi D(1.5D) = \left[\frac{\pi}{2} + 1.5\pi\right] D^2 = k' D^2 \qquad \text{and} \qquad \frac{A_{T2}}{A_{T1}} = \left[\frac{\cancel{k'}}{\cancel{k'}}\right]\left[\frac{D_2}{D_1}\right]^2 = \left[\frac{D_2}{D_1}\right]^2$$

A corollary to Scaling Principle 2 is that any area ratio on geometrically similar figures is a constant. This is observed for our cylinders by the equality of the ratios of A_s and A_T.

Scaling Principle 3 relates volume to length cubed. For the cylinders, we know that the total volume V_T is equal to $\pi D^2 h/4$. Substituting h = 1.5 D, we can show the validity of this principle.

$$V_T = \frac{\pi D^2 h}{4} = \frac{\pi D^2(1.5D)}{4} = \left[\frac{1.5\pi}{4}\right] D^3 = k'' D^3 \qquad \text{and} \qquad \frac{V_{T2}}{V_{T1}} = \left[\frac{\cancel{k''}}{\cancel{k''}}\right]\left[\frac{D_2}{D_1}\right]^3 = \left[\frac{D_2}{D_1}\right]^3 .$$

As before, we may choose any length or volume we wish, and the relation is still valid. Extending the corollary for areas to volumes, the ratio of any two volumes on geometrically similar figures is a constant, and we can relate volumes and areas by substitution. The result is that any volume is proportional to any area raised to the 3/2 power. We use total volume V_T and base area A_s as an example.

$$\frac{V_{T2}}{V_{T1}} = \left[\frac{D_2}{D_1}\right]^3 = \left[\left(\frac{D_2}{D_1}\right)^2\right]^{3/2} = \left[\frac{A_{s2}}{A_{s1}}\right]^{3/2} .$$

The power of scaling is demonstrated in the following example problems.

<u>Scaling Example 3.</u> A tree 10 ft tall has a trunk circumference of 18 in. What is the height of a tree with a 25 in. diameter trunk? (ft)

We don't know anything about the shape of the trees here except that they may be assumed to have the same shape but different sizes. The two lengths we choose are the height h and circumference c. Writing the proportion with h_1 = 10 ft, c_1 = 18 in and c = πD = π(25 in), we quickly obtain the answer.

$$\frac{h}{c} = \frac{h_1}{c_1} \qquad \text{or} \qquad h = \left[\frac{h_1}{c_1}\right] c = \left[\frac{10\ \text{ft}}{18\ \cancel{\text{in}}}\right] \pi(25\ \cancel{\text{in}}) = 43.6\ \text{ft}$$

<u>Scaling Example 4.</u> On the Texas map on the wall, the scale is given as 1:1,580,000. The area of our great State is 267,339 square miles. Find the map area of Texas. (in^2)

The map scale means that the ratio of any length dimension on the map d_m to the actual span on Texas d_{Texas} equals 1/1,580,000. We want the map area A_m, and use the second scaling principle to find it.

$$\frac{A_m}{A_{Texas}} = \left[\frac{d_m}{d_{Texas}}\right]^2 \qquad \text{or} \qquad A_m = \left[\frac{d_m}{d_{Texas}}\right]^2 A_{Texas} = \left[\frac{1}{1{,}580{,}000}\right]^2 267{,}339\ \text{mi}^2 = 1.071\text{x}10^{-7}\ \text{mi}^2$$

We need only to convert the answer to in^2.

$$A_m = 1.071\text{x}10^{-7}\ \cancel{\text{mi}}^2 \left\{\frac{5280\ \cancel{\text{ft}}}{\cancel{\text{mi}}}\right\}^2 \left\{\frac{12\ \text{in}}{\cancel{\text{ft}}}\right\}^2 = 430\ \text{in}^2$$

<u>Scaling Example 5.</u> Four equilateral triangles are drawn connected on a sheet of paper so a pyramid of volume 8 cm^3 can be made by cutting and folding. If the paper is reduced twice at the 74% reduction setting of a copier (e.g., 1 in. → 0.74 in.), what is the volume of the resulting pyramid? (cm^3)

The volume of this pyramid can be calculated using solid geometry, but it is unnecessary to do this. Scaling Principle 3 is used, with V_2 the unknown, $V_1 = 8\ cm^3$, and $x_2/x_1 = (0.74)^2$.

$$\frac{V_2}{V_1} = \left[\frac{x_2}{x_1}\right]^3 \quad \text{or} \quad V_2 = \left[\frac{x_2}{x_1}\right]^3 V_1 = \left[(0.74)^2\right]^3 \; 8\text{ cm}^3 = \; 1.31\text{ cm}^3$$

Scaling Example 6. The largest dinosaurs were the Brachiosaurii, which weighed an estimated 80 tons. Given that the average elephant weighs about 7 tons and stands 10 ft high, estimate the height of a Brachiosaurus. (ft)

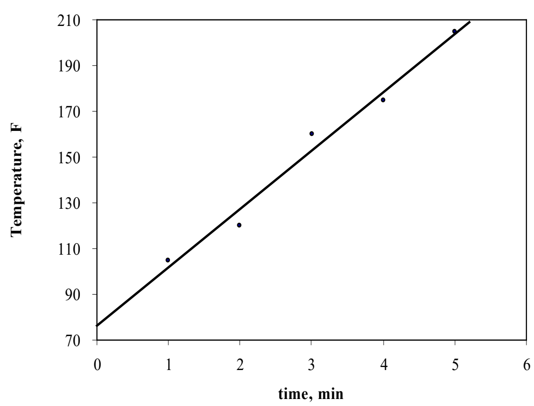
It may seem like a stretch to consider that an elephant and Brachiosaurus have identical shape, but the error is probably not as great as you might expect. We use the weight version of Scaling Principle 3 directly to get an answer.

$$\frac{m_2}{m_1} = \left[\frac{x_2}{x_1}\right]^3 \quad \text{or} \quad \frac{80\ \text{~~tons~~}}{7\ \text{~~tons~~}} = \left[\frac{x_2}{10\text{ ft}}\right]^3, \text{ from which } x_2 = \left[\frac{80}{7}\right]^{1/3} 10\text{ ft} = 22.5\text{ ft}$$

J. Best Fit Lines

If we were given two (x,y) data pairs, we could construct a line that passes through both points. But how do we construct a line that best fits N>2 data pairs which do not lie on a single line? It is not uncommon to do a series of N measurements of the form (x_i,y_i) and then to use these data to predict a "best" value of y at some value of x.

For example, suppose we have an oven that is heating from room temperature to 350ºF. We make temperature measurements every minute and they are (1 min, 105ºF), (2 min, 120ºF), (3 min, 160ºF), (4 min, 175ºF) and (5 min, 205ºF). These (N = 5) data are plotted in the figure which also shows a line that best fits the data. We could use this line to predict, say, the temperature at a time of 3.8 min or even 6 min. The immediate problem is to calculate the slope m and intercept b of this best fit line.



From a theoretical perspective, we can define a deviation of a point (x_i,y_i) from the line y = mx+b as deviation = $\delta_i = y_i - y$. We square the deviation to make all deviations positive, and then we add them up to obtain

$$\sum_i \delta_i^2 = \sum_i (y_i - y)^2$$

We minimize the function with respect to the intercept b by taking the derivative with respect to b and setting the result equal to zero. Simplifying, we obtain $b = \bar{y} - m\bar{x}$ where $\bar{y}$ is the average of the y_i values and $\bar{x}$ is the average of all the x_i values. We next take the derivative with respect to m and set the result equal to zero after substituting the value obtained for b. Upon simplification, the slope m becomes

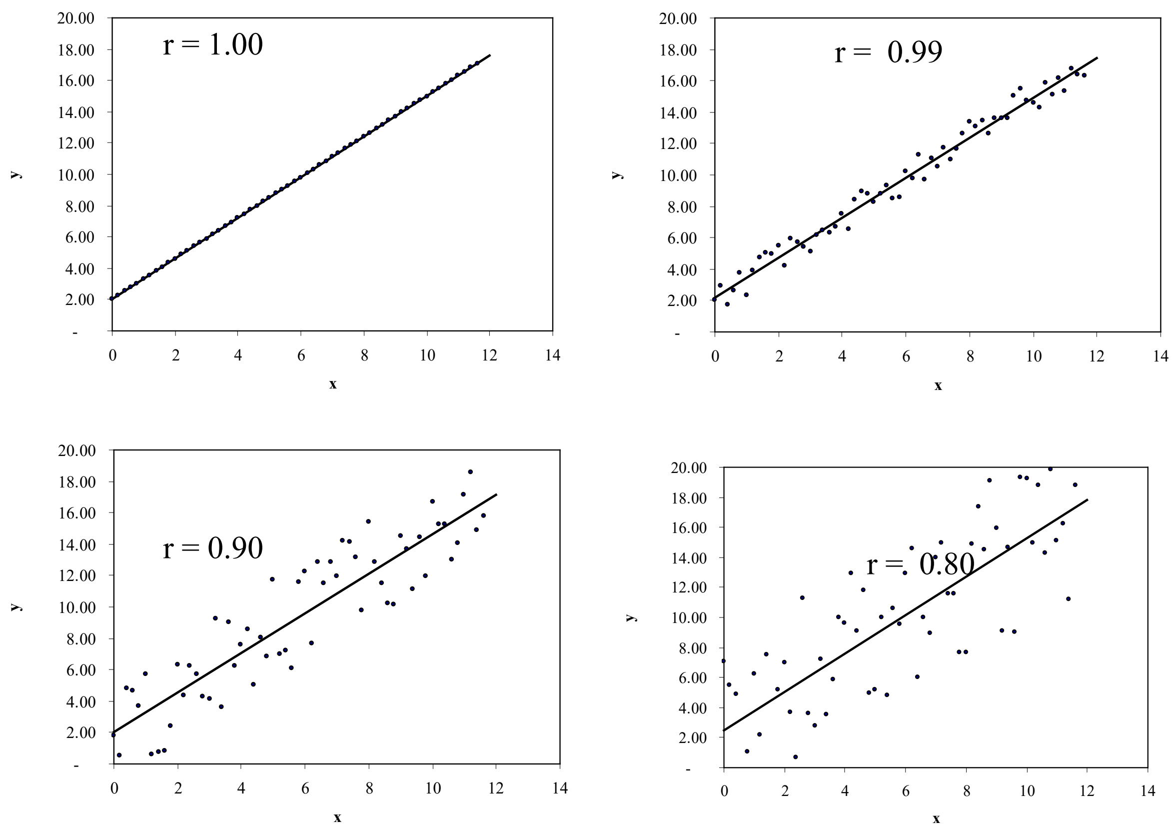
$$m = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}, \text{ and } b = \bar{y} - m\bar{x}\,.$$

These are the values of the slope m and intercept b that minimize the square of the deviation and that constitute a best-fit straight line.

From a practical standpoint, engineering calculators have summation keys for data entry. Generally, the sum registers are cleared using a CLRΣ key (or equivalent; see your calculator operating manual.). Then, data points are entered as (x,y) pairs, and each pair is summed using a Σ+ key (or equivalent). Once all the data are entered, there are keys that will output m and b as well as $\bar{x}$ and $\bar{y}$. Some calculators have a Σ- key (or equivalent) that subtracts a data pair. This is useful when a wrong pair is entered by mistake.

For the temperature problem, the following are results obtained using this method: m = 25.5ºF/min, b = 76.5ºF, $\bar{x}$ = 3 min and $\bar{y}$ = 153ºF. The best estimate of the temperature at t = 3.8 min is 173ºF, and at 6 min, we estimate the temperature to be 230ºF.

The correlation coefficient r is a number between -1 and +1. It is an indication of how close the data points lie on the best fit straight line. A line with no correlation to the data has a slope of zero. The sign of r is the same as the sign of the slope of the best fit line. Calculators have an r key with the $\bar{x}$ and $\bar{y}$ keys. The equation for r



is

$$r = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2 \sum_i (y_i - \bar{y})^2}}$$

The figures show data points scattered to produce an r between 0.80 and 1.0. For the temperature problem, r = 0.990.

<u>Best Fit Line Example 1.</u> What is the best fit slope of the data (1,5), (2,2), (3,-4), (4,-6.5), (5,-8)?

First, the summation registers of the calculator are cleared. Then, each data point is entered and the Σ+ key is pressed. Looking up the slope m after the data points are entered gives the answer, -3.45.

Best Fit Line Example 2. What is the correlation coefficient for the previous example data?

The r key on the calculator shows -0.977.

Best Fit Line Example 3. George tosses a bean bag at 5 ft increments to 30 ft. His measured distances after each toss are 5.2 ft, 11 ft, 13.5 ft, 19 ft, 23.4 ft and 28 ft. What is the y-intercept of the best fit straight line predicting his toss distance? (ft)

The x data are the desired distances (5 ft, 10 ft, 15 ft, etc.) and the y values are listed in the problem. Entering these data and computing the intercept yields 1.01 ft.

It must be realized that we could actually define two best fit lines for given data, and the lines are not coincident. For example, in Best Fit Line Example 1, the slope of the line is –3.45. But if we entered the data as (5,1), (2,2), (-4,3), (-6.5, 4) and (-8,5), our slope becomes –0.276, and the reciprocal is –3.62, NOT –3.45! Therefore, it is important on the UIL contests to take (x,y) data as given and not the reverse. Slopes and intercepts are different, but, interestingly, the correlation coefficient r is the same for both lines. For a problem like Best Fit Line Example 3, only one set of data is given. Are they x or y values? In cases like this, the "exact" values will be by definition x, and the measured or inexact values will be y. For this problem, the (x,y) data pairs are (5 ft, 5.2 ft), (10 ft, 11 ft), (15 ft, 13.5 ft), etc.

K. Problems Involving Matrix Algebra

A matrix is simply an array of numbers. Matrices are bounded by square brackets, and the two most popular matrices are so-called 2x2 matrices like $\begin{bmatrix} 25 & 10 \\ -5 & 60 \end{bmatrix}$ and so-called 3x3 matrices like $\begin{bmatrix} 9 & 5 & -2 \\ 7 & -6 & 1 \\ 0 & 3 & 2 \end{bmatrix}$. A term in a matrix is given by a variable with two subscripts, the first being the row number and the second being the column number. Examples could be

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

The variable representing an entire matrix is the same as the variable for any term in the matrix, in bold and without subscripts. Two matrices with the same number of rows and columns can be added or subtracted by adding or subtracting individual terms. For example,

$$\mathbf{A} + \mathbf{C} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} + C_{11} & A_{12} + C_{12} \\ A_{21} + C_{21} & A_{22} + C_{22} \end{bmatrix}$$

Matrix Example 1. What is the value of C_{23} if $\mathbf{C} = 2\mathbf{A} + 3\mathbf{B}$, and $\mathbf{A} = \begin{bmatrix} 6 & 4 & 9 \\ 5 & 0 & -4 \\ -2 & 1 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 9 & -5 \\ 8 & -9 & 0 \\ 3 & -8 & 6 \end{bmatrix}$?

$C_{23} = 2A_{23} + 3B_{23} = 2(-4)+3(0) = -8.00.$

Two matrices can be multiplied if the number of columns of the first matrix equals the number of rows of the second matrix.

$$\mathbf{AB} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} \\ A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31} \end{bmatrix}$$

A matrix with the same number of rows and columns is called a square matrix. Two square matrices can be multiplied. The products for 2x2 and 3x3 matrices are given as

$$\mathbf{AC} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11}C_{11} + A_{12}C_{21} & A_{11}C_{12} + A_{12}C_{22} \\ A_{21}C_{11} + A_{22}C_{21} & A_{21}C_{12} + A_{22}C_{22} \end{bmatrix}$$

and

$$\mathbf{C} = \mathbf{BD} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} =$$

$$= \begin{bmatrix} B_{11}D_{11} + B_{12}D_{21} + B_{13}D_{31} & B_{11}D_{12} + B_{12}D_{22} + B_{13}D_{32} & B_{11}D_{13} + B_{12}D_{23} + B_{13}D_{33} \\ B_{21}D_{11} + B_{22}D_{21} + B_{23}D_{31} & B_{21}D_{12} + B_{22}D_{22} + B_{23}D_{32} & B_{21}D_{13} + B_{22}D_{23} + B_{23}D_{33} \\ B_{31}D_{11} + B_{32}D_{21} + B_{33}D_{31} & B_{31}D_{12} + B_{32}D_{22} + B_{33}D_{32} & B_{31}D_{13} + B_{32}D_{23} + B_{33}D_{33} \end{bmatrix}$$

This relationship can be condensed as $C_{ij} = \sum_{k=1}^{3} B_{ik}D_{kj}$. Multiplication of matrices is NOT commutative. That is, **AB≠BA**.

<u>Matrix Example 2.</u> What is b if C_{13} =17, **C** = **AB**, and $\mathbf{A} = \begin{bmatrix} b & 4 & 9 \\ 5 & 0 & -4 \\ -2 & 1 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 9 & -5 \\ 8 & -9 & 0 \\ 3 & -8 & 6 \end{bmatrix}$?

$C_{13} =17 = A_{11}B_{13}+A_{12}B_{23}+A_{13}B_{33}$ = (b)(–5)+(4)(0)+(9)(6), from which b = 7.40.

The determinant of a square matrix is a single number. It is written as "Det**B**" or by replacing the square brackets with vertical lines. For a 2x2 matrix, Det**B** is

$$\text{Det}\mathbf{B} = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = B_{11}B_{22} - B_{12}B_{21}$$

For a 3x3 matrix,

$$\text{Det}\mathbf{B} = \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} = B_{11}B_{22}B_{33} + B_{12}B_{23}B_{31} + B_{21}B_{32}B_{13} - B_{11}B_{32}B_{23} - B_{22}B_{31}B_{13} - B_{33}B_{21}B_{12}$$

Matrix Example 3. What is the determinant of the matrix $\mathbf{F} = \begin{bmatrix} 45 & 9 \\ 20 & -34 \end{bmatrix}$?

Det**F** = (45)(–34) – (9)(20) = –1710.

Matrix Example 4. What is the determinant of $\mathbf{A} = \begin{bmatrix} 2 & 4 & 9 \\ 5 & 0 & -4 \\ -2 & 1 & 5 \end{bmatrix}$?

Det**A** = –15.0.

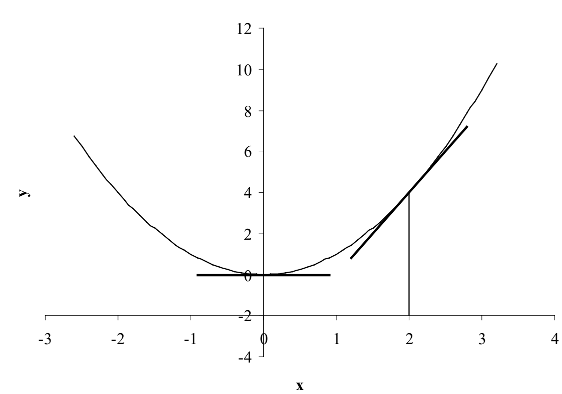
L. Problems Involving Calculus

It is beyond the scope of this contest manual to describe the background information for calculus. A brief introduction to the two main areas of calculus, differentiation and integration, is given with some pointers on problem solving. Special sections on related rates and solids of revolution are also presented. Everything is geared towards application of these methods to problem solving on the Calculator Applications Contest, so do not assume that these sections will substitute for a formal course in calculus. For calculus of trig functions, radian measure is assumed here and on UIL contests unless explicitly stated otherwise.

i. Differential Calculus Problems

We know that the slope of a line of the form y=mx+b is equal to m for all values of x. However, for non-linear functions like $y=5x^3$ or y=3cos(x/3), the function plots as a curve, and the slope of a tangent line to the curve varies depending on the value of x. The figure shows a plot of $y=x^2$ where at x=0, the slope is 0, but at x=2, the slope is 4.

Differential calculus allows us to calculate explicitly the slope of functions. For lines, the slope is traditionally represented by m. For functions, the slope is reminiscent of the rise-over-run formula. The rise is a small ("differential") value of y we might represent as Δy, and the run is a differential value Δx. The slope then could be represented as Δy/ Δx. However, we want Δy and Δx to be very, very small; in fact, we want them to shrink towards zero. The representation of this idea is to replace the "Δ" with a "d", making Δy→dy and Δx→dx. The slope then is represented as dy/dx. This nomenclature is confusing until one gets used to the idea. At this point, it is emphasized that the d's are not variables but are associated with the x and y variables.



Getting the slope of a function y is obtained by applying "d()/dx" to both sides of the equation. When we do this, we say we are taking the derivative of y with respect to x. For example, for $y=x^2$, we could write:

$d(y)/dx=d(x^2)/dx$. For the right side of the equation, there are a lot of rules which are summarized in Appendix I. The rule we need is $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$. Substituting x for u, and 2 for n, we get $\frac{d(x^2)}{dx} = 2x^{2-1}\frac{dx}{dx}$ which simplifies to 2x since dx/dx = 1. The full relationship becomes dy/dx=2x, meaning that the slope of a tangent line on the curve $y=x^2$ is equal to 2x. When x = 0, dy/dx = 2(0) =0, and when x=2, dy/dx=2(2)=4.

Calculus Example 1. What is the slope of the function $y = x^2 \sin(4x)$ at x = 1.5?

The derivative of a product is $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. Using this rule, the answer is

$$\frac{d(x^2 \sin(4x))}{dx} = 2x\sin(4x) + x^2(\cos(4x))4 = 2x\sin(4x) + 4x^2\cos(4x)$$

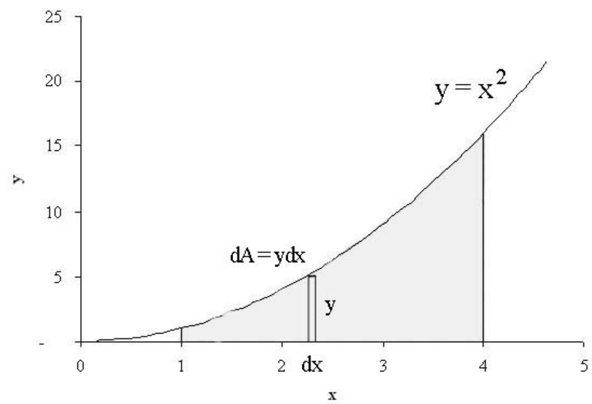
Evaluated at x = 1.5, the slope is 7.80.

Calculus Example 2. At what value of x does the slope of the function $f(x) = 5x^2 - 13x + 85$ equal zero?

The slope is the derivative:

$$\frac{df}{dx} = 10x - 13$$

Setting the derivative equal to zero yields the answer, x = 13/10 = 1.30.



ii. Integral Calculus

In some sense, integral calculus is the reverse of differential calculus. That is, if the derivative is dy/dx=2x, then the integral produces $y=x^2+C$, where C is a constant. By studying the integrals in the appendix and comparing them to derivatives in the appendix, you can see the similarities. The basic concept of an integral though is the summation of differential elements over some range of x.

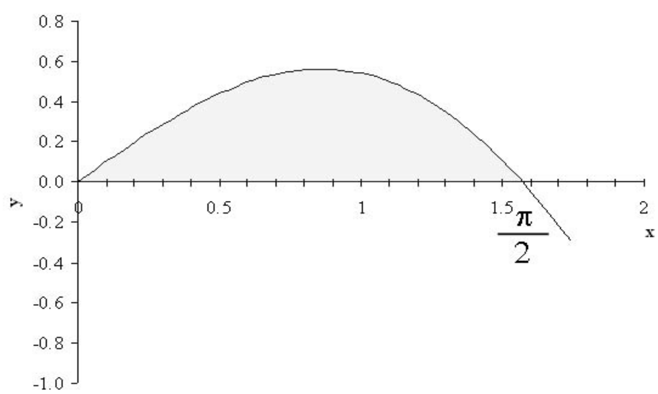
When the differential element is an area, we can compute the areas bounded by curves. For example, consider the figure showing a plot of $y=x^2$. Suppose we wanted to know the area between the x axis and the curve bounded by the lines $x_1=1$ and $x_2=4$. Armed with the knowledge that the area of a rectangle is base times height, we could divide the area into n rectangles of height $y = x^2$ and base $\Delta x = (x_2-x_1)/n$. The sum of these n rectangles is given as $A = \sum_{x_1}^{x_2}(x^2)\left(\frac{x_2 - x_1}{n}\right) = \sum_{x_1}^{x_2}(x^2)\Delta x$, where Δx is the base of each rectangle. If we let n→∞, then the area rectangles become vanishingly small, and the base goes from Δx to dx as described in the previous section. The summation sign Σ is changed to an integral sign ∫ to reflect the differential nature of the problem, and the summation becomes $A = \int_{x_1}^{x_2} x^2 dx$. Since we are summing an infinite number of differential rectangles, the error in area for each rectangle becomes vanishingly small, and the summation (or integration) yields an *exact* value for the area, not an approximation. The appendix lists values for various integrals. The ones that apply here are

$\int u^n du = \frac{u^{1+n}}{1+n} + C$ and $\int_{x_1}^{x_2} du = u(x_2) - u(x_1)$. Combining the two with u=x and n=2, we obtain $A = \int_{x_1}^{x_2} x^2 dx = \frac{1}{3}\left(x_2^3\right) - \frac{1}{3}\left(x_1^3\right)$. Substituting x_1=1 and x_2=4, we calculate the area under the curve to be $A = \frac{1}{3}\left(4^3\right) - \frac{1}{3}\left(1^3\right) = 21$.

Calculus Example3. What is the area under the curve of the function y = 3sin(5x) bounded by x = 0 and x = π/5? The form of the integral is

$$\text{A} = \int_0^{\pi/5} 3\sin(5\text{x})\text{dx} = \frac{3}{5}\int_0^{\pi/5} \sin(5\text{x})(5\text{dx}) = \frac{3}{5}\left[-\cos(5\text{x})\right]_0^{\pi/5}$$

$$\text{A} = \frac{3}{5}\left[-\cos(\pi) + \cos(0)\right] = \frac{3}{5}(2) = 1.20$$



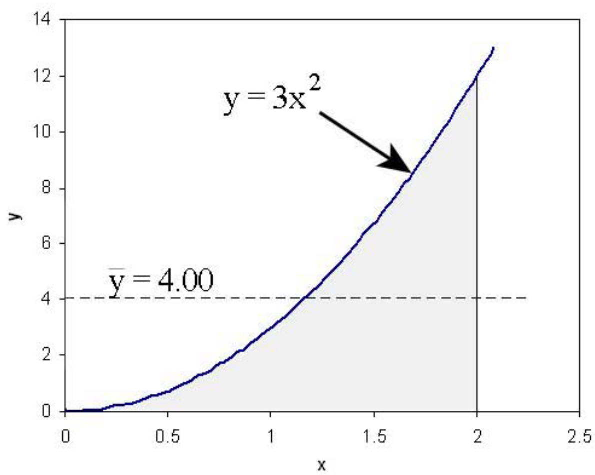
Calculus Example 4. What is $\int_0^{\pi/2} x\cos x dx$? The solution is given in the appendix for the function xcosx, but it is derived here. We can integrate cosx to yield sinx, but the first x in the integral gets in the way. A solution can be obtained using the integral of a product rule, listed in the appendix in the form $\int u dv = uv - \int v du$. By letting u=x, v=sinx, we see that du=dx and dv=(cosx)dx. Substituting into the equation yields $\int x\cos x dx = (x)(\sin x) - \int (\sin x) dx$

which simplifies to

$$\int x\cos x dx = x\sin x + \cos x + \text{C}$$

Including the bounds 0 and π/2, we obtain

$$.\int_0^{\pi/2} x\cos x dx = \left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - \left((0)\sin 0 + \cos 0\right) = \left(\frac{\pi}{2}\right) - (1) = 0.571$$



Given an area like those shown in the previous figures, it is possible to calculate the average value of the ordinate or y value by dividing the area by the difference in the bounds of integration, $x_2 - x_1$. The average value of y is also called the mean value of y and is represented mathematically by a bar over the variable, $\overline{y}$ (don't confuse this with vector nomenclature!). Formally, given a function y = f(x) defining an area between two x values, x_1 and x_2, the average value of y is given as

$$\overline{y} = \frac{\int_{x_1}^{x_2} y dx}{\int_{x_1}^{x_2} dx} = \frac{\int_{x_1}^{x_2} y dx}{x_2 - x_1}$$

Calculus Example 5. What is the mean value of y for the area under the curve $y = 3x^2$ between 0 and 2?

Confirmed on the above graph, we do the integration to solve for $\overline{y}$:

$$\overline{y} = \frac{\int_{x_1}^{x_2} y dx}{\int_{x_1}^{x_2} dx} = \frac{\int_{0}^{2} 3x^2 dx}{\int_{0_1}^{2} dx} = \frac{x^3\Big|_0^2}{2-0} = \frac{8-0}{2} = 4.00$$

iii. Related Rates

There is a class of differential calculus problems called related rates problems. These problems involve objects that change rate. For example, consider a board leaning against a wall. If it becomes unstable and slides to the floor, locations of the board accelerate and move at different but related rates. Related rate problems use methods to find these rates. As the board falls, it rotates, and the rotation is also a rate that can be related. We are relating some type of velocity to another velocity, hence, the term *related rates*.

For lengths, the solution is to write a geometric equation relating the two lengths and then to take a time derivative d/dt. Since the rate of change of length with respect to time is a velocity, we obtain an equation that relates the velocities.

Calculus Example 6. A 12-ft long ladder is leaning against a tall wall with the base 4 ft from the wall. It starts to slip with the ladder end contacting the wall moving vertically downward at a velocity of 1 ft/s. At what positive horizontal velocity does the other end of the ladder move along the floor?

For the ladder problem, we know that x, the distance from the wall to the floor-end of the ladder, is related to y, the distance from the floor to the wall-end of the ladder, by the equation

$$x^2 + y^2 = (12\text{ft})^2 .$$

We know that x = 4 ft, and from this equation, y = 11.3 ft. The y velocity is –1ft/s since the ladder tip is moving downward to decrease y as time progresses. Taking the time derivative of both sides of the length equation, d/dt produces

$$d\left(x^2 + y^2\right)/dt = d\left((12\text{ft})^2\right)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

A convention for time derivatives is to represent "d/dt" by a dot over the variable. This is adopted here to produce a solution for dx/dt= $\dot{x}$:

$$\dot{x} = -\frac{y}{x}\dot{y} = -\frac{11.3\text{ft}}{4\text{ft}}(-1\text{ft}/\text{s}) = 2.83\text{ft}/\text{s}$$

Calculus Example 7. A spotlight rotates 360 degrees in 1 minute and is 30 feet away from a long wall. At what distance from the spotlight to the light spot on the wall is the light spot on the wall moving with a velocity of 10 mph? (ft)

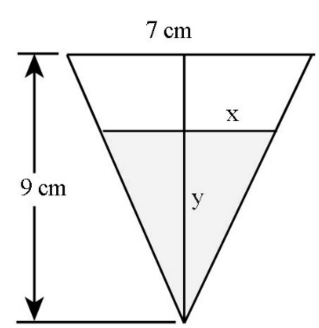
The motion of the light on the wall is related to the rotational speed of the spotlight. For the spotlight problem, we must first write an equation for x where x is measured along the wall at the closest approach to the spotlight. The rotational speed of the spotlight is 2π radians in 1 minute, or $\dot{\theta}$=2π radians/min. The direction of rotation is not important, so we'll set it to start at θ=0 rad when t=0, as shown in the figure. The value of x is given by x = (30 ft)tanθ. By taking the time derivative of this equation, we get

$$\dot{x} = \left[(30\text{ft})\sec^2\theta\right]\dot{\theta}$$

Here, $\dot{x}$=10 mph = 880 ft/min and $\dot{\theta}$=2π radians/min. Solving for θ gives θ = 1.09 rad. The value of x is ±57.5 ft, so the distance z from the spotlight to either spot on the wall is $z = \sqrt{(57.5\text{ft})^2 + (30\text{ft})^2} = 64.8\text{ft}$.

Related rate problems can also treat changes in areas and volumes with time. For these problems, we start with an equation for the area or volume and take a time derivative. For area, imagine water is poured on a level surface to make a circular puddle. If the puddle radius increases at 5 in/s when the puddle diameter is 2 ft, what is the rate (in^2/s) at which the circle area is increasing? The area of the puddle is given by $A = \pi R^2$, so taking time derivatives produces $\dot{A} = 2\pi R\dot{R}$. Here R = 1 ft and $\dot{R}$ = 5 in/s. Substituting, we obtain $\dot{A}$ = 377 in^2/s.

Calculus Example 8. Old-fashioned water coolers have conically shaped paper cups. One such cup is 9 cm tall with a maximum diameter of 7 cm at the top. If the cup is filled with water at a rate of 30 cm^3/s, at what water level measured from the cup bottom (cm) does the water level change at 1 cm/s? The figure shows a cross section of the cup, revealing that x/y = 3.5/9 or x = 3.5y/9. The volume of water in the cup V_w is given by



$$V_w = \frac{\pi}{3}x^2 y = \frac{\pi}{3}\left(\frac{3.5y}{9}\right)^2 y = \frac{\pi}{3}\left(\frac{3.5}{9}\right)^2 y^3$$

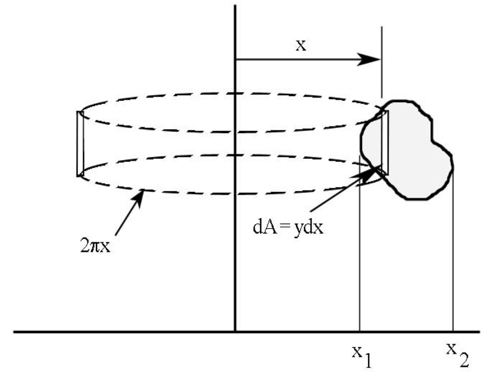
Taking time derivatives, we get

$$\dot{V}_w = \pi\left(\frac{3.5}{9}\right)^2 y^2 \dot{y}$$

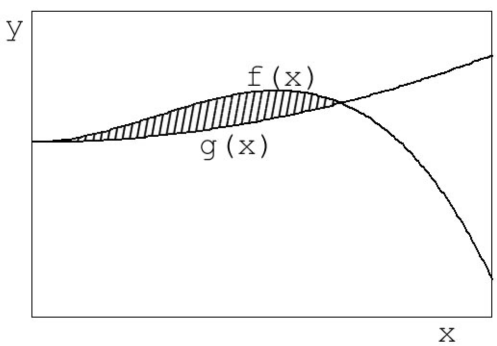
For $\dot{V}_w$ = 30 cm^3/s and $\dot{y}$ = 1 cm/s, we can solve the equation for y and obtain y = 7.95 cm.

iv. Solids of Revolution

A solid of revolution is a solid geometric object created by sweeping a closed area around an axis, as shown in the figure for a rotation of the shaded area around the y axis. For the UIL Calculator Applications Contest, rotations are allowed around the x axis, the y axis, and any arbitrary line parallel to either the x axis or y axis. The volume may be computed using integral calculus.



Consider the area in the figure bound by two functions f(x) and g(x). Here, we define f(x)≥g(x) in the area of interest. The hatched, bound area may be rotated about any axis parallel to the x axis or y axis outside the bound area to create a solid of revolution. The equation describing these axes may be given as y = b (parallel to the x axis) and x = a (parallel to the y axis), respectively, where a and b are constants.



There are two methods to describe the volume of a solid of revolution: the shell and disc methods. The "shell" and "disc" shapes describe the differential volume element for each method. For the shell method, the differential volume element is a thin cylindrical sleeve (or shell) as shown on the figure. For the disc method, the differential volume element is a thin, potentially hollow disc. For the Calculator Applications Contest, the differential area element swept around an axis of revolution may always be taken to be [f(x)-g(x)] tall and δx thick. In this case, the disc method is used for rotations parallel to the x-axis and the shell method is used for rotations parallel to the y-axis. The sign of the volume depends on the choice of f(x) and g(x) for the bounding curves. In practice, the negative sign for volume – if it occurs – may be ignored.

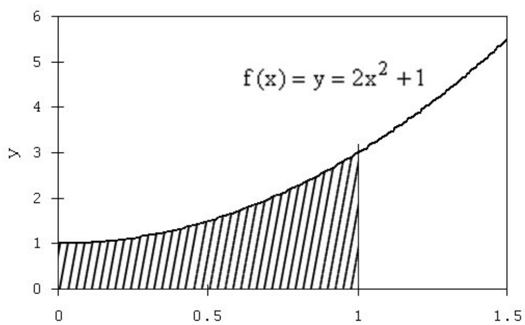
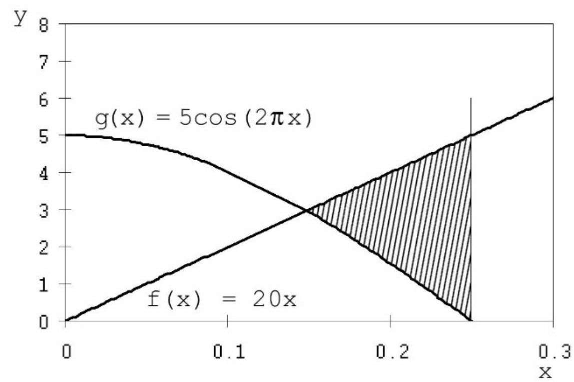
Rotating Parallel to the x-axis (y = b, *Disc Method*)

$$V = \pi \int_{x_0}^{x_1} \left\{ [f(x) - b]^2 - [g(x) - b]^2 \right\} dx$$

Rotating Parallel to the y-axis (x = a, *Shell Method*)

$$V = 2\pi \int_{x_0}^{x_1} (x - a)\{f(x) - g(x)\} dx$$

The axis of revolution for solids of revolution will be given in the title bar of the geometry problem, as shown in the following two examples.

Calculus Example 9.	Calculus Example 10.
SOLID OF REVOLUTION (Axis of Revolution: y = –3)	SOLID OF REVOLUTION (RAD) (Axis of Revolution: x = 0.1)
	
Volume = ?	Volume = ?
Answer = ________	Answer = _________

<u>Calculus Example 9 (Rotation Parallel to the x-axis)</u>. First, write f(x) and g(x). Here, $f(x) = y = 2x^2 + 1$, g(x) = 0, and the axis of rotation is y = –3. Using the Disc Method,

$$V = \pi \int_{x_0}^{x_1} \left([f(x) - b]^2 - [g(x) - b]^2 \right) dx$$

$$V = \pi \int_0^1 \left([(2x^2 + 1) + 3]^2 - [0 + 3]^2 \right) dx$$

$$V = \pi \int_0^1 \left(4x^4 + 16x^2 + 7 \right) dx$$

$$V = \pi \left(\frac{4x^5}{5} + \frac{16x^3}{3} + 7x \right) \Bigg|_0^1 = 13\frac{2}{15}\pi = 41.3$$

<u>Calculus Example 10 (Rotation Parallel to the y-axis)</u>. In this example, f(x) = 20x and g(x) = 5cos(2πx). Rotation is parallel to the y-axis along x = 0.1. The bounds of integration are given as $x_0 = 0.1487$ (using the solver function on the calculator) and $x_1 = 0.25$ (where g(x) = 0). Using the Shell Method,

$$V = 2\pi \int_{x_0}^{x_1} (x - a)\{f(x) - g(x)\} dx$$

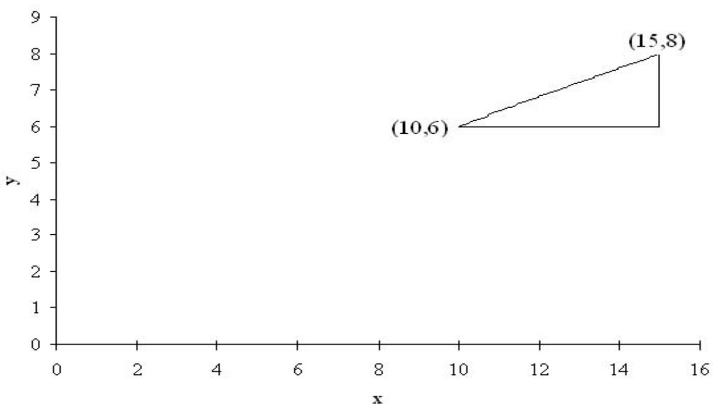
$$V = 2\pi \int_{0.1487}^{0.25} (x - 0.1)\{20x - 5\cos(2\pi x)\}dx$$

$$V = 2\pi \int_{0.1487}^{0.25} \left(20x^2 - 2x - 5x\cos(2\pi x) + 0.5\cos(2\pi x)\right)dx$$

$$V = 2\pi\left[\frac{20x^3}{3} - x^2 - \frac{5}{4\pi^2}\left(\cos(2\pi x) + 2\pi x\sin(2\pi x)\right) + \frac{0.5}{2\pi}\sin(2\pi x)\right]_{0.1487}^{0.25}$$

$$V = 2\pi\left(-0.0777 - (-0.107)\right) = 0.182$$

Calculus Example 11. What is the volume of a solid of revolution about the y axis of a right triangle defined by the points, (10,6), (15,6) and (15,8)?



The right triangle is shown in the figure. From the two data points, the equation of the slant line is $f(x) = \frac{2}{5}x + 2$. The function g(x) = y – 6, and the equation for the axis of rotation is x = 0. The volume of the solid of revolution is given by the shell method:

$$V = 2\pi \int_{10}^{15} (x - 0)\left[\left(\frac{2}{5}x + 2\right) - 6\right]dx$$

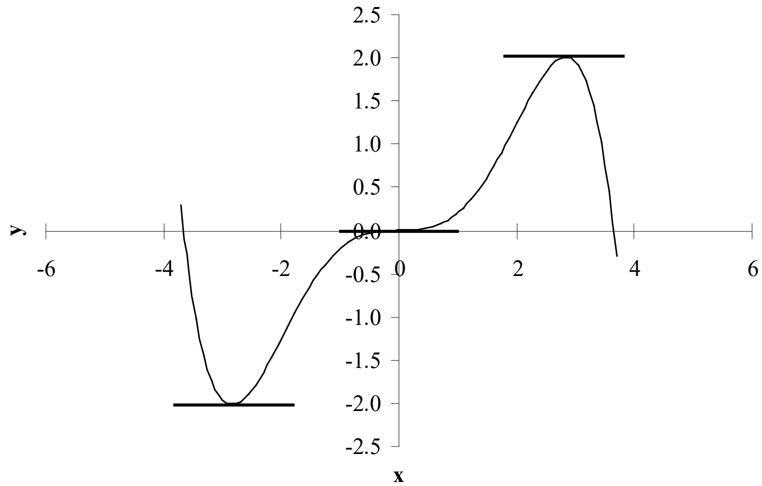
$$V = 2\pi \int_{10}^{15} x\left[\frac{2}{5}x - 4\right]dx$$

$$V = 2\pi \int_{10}^{15}\left[\frac{2}{5}x^2 - 4x\right]dx = 2\pi\left(\frac{2}{15}x^3 - 2x^2\right)\Bigg|_{10}^{15} = 419.$$

It is speculated that most contestants will not directly integrate the equations as shown in the solution to the examples above, but they will rather use integration capabilities of the calculator to enter the integrand and bounds, then letting the calculator compute the value for the integral.

v. Maxima and Minima

As described in Section Li of this chapter, the derivative of an x-y plot, dy/dx, gives the slope of a curve at a point. If the slope equals zero, then one of three possibilities exists. First, as shown in the figure, the slope is zero when y is a minimum, at $x = -\sqrt{8} = -2.82$. Second, it could occur when y is at a maximum as when $x = +\sqrt{8}$. Third, y could be at an inflection point, where y is a maximum on one side and a minimum on the other side; this occurs on the plot when x = 0. Finding these extreme points (maxima, minima, inflections) is done by doing dy/dx, also called a first derivative, setting it equal to zero and solving for x. Identifying an extreme point by type requires evaluation of the second derivative, denoted by d^2y/dx^2. The second derivative is simply the derivative of the derivative. If the second derivative is positive at the extreme point, the point is a minimum. If the second derivative is negative, the point is a maximum. If the second derivative is zero, the point is an inflection.



Calculus Example 12. The plot above is given by the equation $y = -\frac{x^5}{60} + \frac{2}{9}x^3$. Find the data points with zero slopes and evaluate the second derivative at those points.

This is not the kind of problem that would appear on a contest since multiple answers are requested. A request for one such item of information is possible though. We begin by taking the first derivative, setting it equal to zero and solving for x.

$$\frac{dy}{dx} = -\frac{x^4}{12} + \frac{2}{3}x^2 = 0$$

This factors as

$$-\frac{1}{12}(x^2)(x - \sqrt{8})(x + \sqrt{8}) = 0$$

Therefore, $x = 0.00$, $\pm\sqrt{8} = \pm 2.83$. This is in agreement with the location of the zero slopes on the plot. Subbing into the original equation, the data points are: ($-\sqrt{8}$,-2.011), (0,0) and ($\sqrt{8}$,2.011). Moving to the second derivative,

$$\frac{d^2y}{dx^2} = -\frac{x^3}{3} + \frac{4}{3}x = -\frac{x}{3}(x+2)(x-2)$$

The second derivative equals +3.77 when $x = -\sqrt{8}$, signifying that the point on the curve is a minimum. The second derivative equals zero when x = 0, signifying an inflection point. For $x = \sqrt{8}$, the second derivative is -3.77, consistent with a maximum.

Calculus Example 13. A farmer has 400 ft of fence and wants to build a rectangular enclosure composed of three equal areas partitioned parallel to one edge. What is the ratio of the rectangle side dimensions that maximizes the area, a number greater than 1?

The width x and length y define the fence perimeter to be (2x + 2y) + 2x = 400 ft, recalling that the partitions within the enclosure require two extra fences of length x. Solving for y, y = 200 ft – 2x. The area of the enclosure A is $A = xy = x(200\text{ ft} - 2x) = (200\text{ft})x - 2x^2$. The area is maximized where the first derivative with respect to x is zero.

$$\frac{dA}{dx} = 200\text{ft} - 4x = 0 \qquad \text{or} \qquad x = 50\text{ ft}$$

That this is a maximum is given by evaluating the second derivative:

$$\frac{d^2A}{dx^2} = \frac{d(200\text{ft} - 4x)}{dx} = -4$$

The negative sign denotes that the point is in fact a maximum. The solution to the problem is the ratio of lengths greater than one, and, since x = 50 ft, y = 200 ft – 2x = 100 ft. The ratio then is y/x = 100ft/50ft = 2.00.

Calculus Example 14. Consider a rectangle with a perimeter of 19 inches. What is the maximum volume of a cylinder created by rotating the rectangle about one of its edges? (in^3)

Let one edge of the rectangle be R and one edge be h. They are related by 2R + 2h = 19 in, or h = 9.5 in - R. By rotating about one of the h edges, we create a cylinder with radius R and height h. The volume is

$$V = \pi R^2 h = \pi R^2(9.5\text{in} - R) = \pi(9.5\text{in})R^2 - \pi R^3$$

We take the first derivative of V with respect to R, set it equal to zero and solve for R:

$$\frac{dV}{dR} = 0 = \pi(19\text{in})R - 3\pi R^2 \quad \text{or} \quad R = 0,\ 19/3\text{ in}$$

The second derivative is

$$\frac{d^2V}{dR^2} = \pi(19\text{in}) - 6\pi R$$

The second derivative is positive when R = 0, so this is a minimum. The second derivative equals –π(19in) when R = 19/3 in, so this is a maximum. The answer is the volume, so

$$V = \pi(19/3\text{in})^2(9.5\text{in} - 19/3\text{in}) = 399\text{in}^3$$

Calculus Example 15. One car is 20 mi south of Point A and travels north at 30 mph. Another car is 45 mi east of Point A and travels west at 45 mph. If they start at the same time, what is their closest approach to each other? (mi)

The equations of motion are for Car1, $x_1 = 0$ and $y_1 = -20\text{mi} + 30\text{mph}(t)$; for Car 2 it is $x_2 = 45\text{mi} - 45\text{mph}(t)$ and $y_2 = 0$. Taking distance in miles and time in hours, the distance X between the two is

$$X = \sqrt{x_2^2 + y_1^2} = \sqrt{(45 - 45t)^2 + (-20 + 30t)^2} = \sqrt{2925t^2 - 5250t + 2425} = \sqrt{u(t)}$$

The first derivative is set equal to zero:

$$\frac{dX}{dt} = 0 = \frac{1}{2} u^{-1/2} (5850t - 5250)$$

There are three terms on the right hand side of this equation, and at least one must equal zero for the product to equal zero. The first, ½, cannot equal zero. The second, $u^{-1/2}$, cannot equal zero, because if it does, the u would be infinite, making X infinite. Therefore, the third term must be zero, making t = 5250/5850 hr = 0.897 hr. The distance between the cars at this time is X = 8.32 mi.

Calculus Example 16. What is the x value of the inflection point for the equation $y = 7x^3 - 33x^2 + 19x - 200$?

The inflection point occurs where the second derivative is zero.

$$\frac{dy}{dx} = 21x^2 - 66x + 19 \quad \text{and} \quad \frac{d^2y}{dx^2} = 42x - 66 = 0$$

Solving, x = 66/42 = 1.57.

M. "Special" Stated Problems: Exceptions to the Three Significant Digit Rule

With three exceptions, every answer written on a UIL Calculator Applications test comes from a straightforward calculation and is written with three-significant-digit accuracy as described in Chapter 2D. These three exceptions are all stated problems and are called "special" stated problems. They are: integer problems, dollar-sign problems, and significant-digit problems. The problem statement and/or answer blank carry indicators signaling you that the problem is special. While the justification for each is dealt with in each subsection, for all these classes of problem, the motivation arises from the fact that they represent actual situations we encounter in the real world.

i. Integer Problems

There are many entities which can't be split into pieces or they cease to exist. Integer numbers for example are no longer integers if some fraction of the number "1" is taken away. Other common examples are a person, a year like 1975, the number of times some action occurred, the number of sides on a polygon, the number of pizzas bought at a store, the number of hot dogs in a package, a basketball game score, etc.

It is possible to talk in summary terms about fractional items. We can say that on average a teenager can eat 2.7 pizzas at one sitting, or that a football team averages 24.6 points per game, but obviously that doesn't mean that we can go to a restaurant and order 2.7 pizzas, or that the team can score 24.6 points in a game.

Integer problems on the UIL Calculator Contest are indicated by the word, "integer", in the answer blank. The answer must be written to an integer or "one's place", independent of how many significant digits are involved. Another way to say this is to say that the fractional part of the answer is omitted by proper rounding. The two example problems illustrate the concept.

Integer Example 1. A copier has two enlargement settings, 1.21 and 1.42 times magnification. Greater enlargements can be obtained by copying an enlarged copy at one of these settings. What is the minimum number of copies needed to double the linear dimension of an image? (integer)

This is a function problem since we need to write an equation for the linear copy size X as a function of the original linear copy size X_0 and the number of enlargements n. Since we want the minimum number of enlargements, we should ignore the 1.21 setting altogether. The equation becomes

$$X = X_0(1.42)^n$$

The ratio of X to X_0 is 2. We solve for n.

$$\frac{X}{X_0} = 2 = (1.42)^n \qquad \text{or} \qquad \text{Log}(2) = n\,\text{Log}(1.42)$$

$$n = \frac{\text{Log}(2)}{\text{Log}(1.42)} = 1.977$$

It's not possible to make 1.977 copies on a copier, so we round the fractional part up to yield the answer.

$$n = 2$$

Again, it doesn't matter that the answer has only one significant digit. It simply must be written to the appropriate integer value.

Integer Example 2. The sum of five consecutive odd integers is 5^3. Find the product of the five numbers. (integer)

This is a translation problem. Consecutive integer problems can be solved by approximating them to be almost equal, getting a solution quickly and checking the answer. Let x equal the middle of the five integers. Then $5x \approx 125$ or $x \approx 25$. We have an answer. Let's check it. Does 21+23+25+27+29 = 125? Yes. We now proceed. The product is the answer, (21)(23)(25)(27)(29) = 9,454,725. This is the answer, and all 7 digits must be written in the answer blank.

ii. Dollar-Sign Problems

Many stated problems deal with dollars and cents. In reality, we usually write a number representing US currency in the form \$d.Cc, where d is integer dollars and Cc represents the integer number of pennies. Dollar-sign problems all have answers which are numbers written in this "dollar and cents" convention, the answer being written to the nearest penny. As for integer problems, the number of significant digits in the answer doesn't matter. Dollar-sign problems are denoted by a "\$" in the answer blank near the "=" sign.

There are two common errors which contestants make relative to dollar-sign problems. The first is failing to write the cents part of a problem when the answer comes out as exact dollars. That is, contestants sometimes write incorrectly "600" instead of "600.00" as an answer to a dollar-sign problem. The second common error occurs when we ask for an answer in units of cents (¢). Unless the problem is an integer problem, the answer should be given to the usual three significant digits, even if this entails fractional cents. The point here is that a "¢" problem is not a "\$" problem.

Dollar-Sign Example 1: A local civic group raised money by going to a shopping mall and asking for quarters. They hoped to lay them flat on the floor touching with a goal of collecting "one mile of quarters". If they stopped just after one mile was exceeded, and if a quarter's diameter is 61/64 inch, how much money did they collect? (\$)

This is a unit conversion problem. The number of quarters N is based on how many "61/64 in" there are in a mile.

$$N = 1\,\not{mi}\left\{\frac{1\text{ Quarter}}{61/64\ \not{in}}\right\}\left\{\frac{12\ \not{in}}{1\ \not{ft}}\right\}\left\{\frac{5280\ \not{ft}}{1\ \not{mi}}\right\} = 66{,}476.066\text{ Quarters}$$

Since they stopped just after one mile was exceeded, they needed the next highest integer number of quarters, or 66,477 quarters. The value of this many quarters is the answer A when written to the nearest penny.

$$A = 0.25\,N = 0.25\,(66{,}477) = \$16{,}619.25$$

<u>Dollar-Sign Example 2.</u> Juice costs $2 per gallon, but ice is cheap. When they fill the 44 ounce cup with ice and add the juice, 60% of the volume is ice. What does the juice in the drink cost the merchant? (¢)

This is also a unit conversion problem. The amount of juice in one drink is 0.4(44 oz) = 17.6 oz. We convert this to gallons and multiply by the cost per gallon as if it were also a conversion factor. Remember that the answer is requested in terms of cents, not dollars, so this is <u>not</u> a dollar-sign problem.

$$A = 17.6\ \cancel{oz} \left\{\frac{1\ \cancel{gal}}{128\ \cancel{oz}}\right\} \left\{\frac{\cancel{\$}2}{1\ \cancel{gal}}\right\} \left\{\frac{100\ ¢}{\cancel{\$}1}\right\} = 27.5\ ¢$$

We note that even though it doesn't make sense to split a penny into a fractional part, it is meaningful here to speak of 27.5¢. The reason is that the merchant sells lots of drinks and buys lots of drink mix. To figure total cost or profit/loss of many sales on a "per cup" basis, the drink cost of 27.5¢ is both allowable and meaningful. In essence, we are speaking here of an average value, so the answer need not be an integer.

<u>Dollar-Sign Example 3.</u> A notepad manufacturer currently makes 125-sheet, 3-inch square pads from rolls of 300-ft long, 36-inch wide paper. The pad retail price is 39 cents. He decides to make smaller, 2.769-inch pads with 115 sheets. Assuming no sheets are scrapped, what is the minimum he can charge for a smaller pad to increase his profits by at least $6.50 per roll of paper? ($)

This is a complicated translation problem. We work the problem on the basis of one roll of paper and assume that the manufacturer buys many rolls of paper to make lots and lots of pads. The implication of this is that while there can be no fractional sheets from one roll of paper, there can be fractional pads. The first thing to calculate is the number of pads N_0 currently obtained from a roll of paper. The sheets are 3 in on a side, so he can get 36/3 = 12 sheets along the width dimension with no loss. The length of a roll is 300 ft or 3600 in. He can get 3600/3 = 1200 sheets along the length of one roll, again without any paper loss. The total number of sheets in one roll is 1200x12 = 14,400 sheets per roll. The value N_0 is obtained using unit conversions.

$$N_0 = \frac{14{,}400\ \cancel{sheets}}{roll} \left\{\frac{1\ pad}{125\ \cancel{sheets}}\right\} = 115.2\ \frac{pads}{roll}$$

The gross income from one roll I_0 is currently

$$I_0 = N_0\ (\$0.39/pad) = 115.2\ \frac{\cancel{pads}}{roll} \left[\frac{\$0.39}{1\ \cancel{pad}}\right] = \frac{\$44.9280}{roll}$$

We write the income to a fractional penny since this is the average income over many rolls of paper. The manufacturer wants to increase his profits by $6.50 per roll, so his new income on one roll I_n must be $44.9280 + $6.50 = $51.4280/roll. The number of pads per roll N_n is obtained the same way N_0 was. The new sheets are 2.769 in on a side, so there are 36/2.769 = 13 sheets obtained along the width dimension of a roll (with a minuscule surplus) and 3600/2.769 = 1300 sheets obtained along the length of a roll (with 0.108 of one sheet loss, or 0.30 in). The number of sheets per roll is 1300x13 = 16,900 sheets/roll, and N_n is calculated, recalling that the new pads have only 115 sheets.

$$N_n = \frac{16{,}900\ \cancel{sheets}}{roll} \left\{\frac{1\ pad}{115\ \cancel{sheets}}\right\} = 146.957\ \frac{pads}{roll}$$

The cost of one pad under the proposed scenario is just I_n/N_n.

$$\frac{I_n}{N_n} = \frac{\dfrac{\$51.4280}{\cancel{roll}}}{146.957\,\dfrac{pads}{\cancel{roll}}} = \$0.3499/pad$$

The answer then is $0.35, answered to the nearest penny.

iii. Significant-Digit Problems

Much engineering and scientific work requires measurements and involves calculations with numbers resulting from measurements. As we discussed in Chapter 2D, measured numbers of necessity have limited accuracy. Measurements are inherently inexact, or uncertain to some degree. Hence arises the need to express measured numbers so as to disclose their accuracy and the corresponding need to develop methods for calculating with inexact numbers so as to retain, but not exaggerate, the accuracy of the results. The method of least significant digits is one of several techniques which satisfy this need. It is, in fact, the simplest method, and as such, provides an introduction to this important aspect of applied mathematics.

In this section, we first describe how significant-digit problems are identified and scored, and how we present inexact numbers in stated problems on the tests. We next review the conventions for writing numbers of limited accuracy and develop a rationale for calculating with such numbers, including multiplication, division, addition, subtraction, powers, roots, and trigonometric functions. In the process of doing this, we develop the concept of a fractional significant digit. The paradox of repeating inexact numbers in a calculation is then presented. Last, we summarize the main concepts of significant-digit problem solving for the reader who is more interested in learning just what is necessary to work these types of problems than in the underlying principles.

<u>Format and Scoring.</u> Significant-digit problems are indicated by having one or more underlined numbers. Underlining signals an inexact number, and numbers in significant-digit problem statements which are not underlined are to be considered exact. Significant-digit problems are also indicated with "SD" in the answer blank, along with the units of the answer.

Scoring of significant-digit problems is like all other problems: +5 for a correct answer and –2 for incorrect, where "correct" in this case also includes the proper number of significant digits. However, if a problem is worked correctly, but the answer is given with the incorrect number of significant digits, the score is +3 for that problem. Here are a few details:

1. If the answer is given with the correct number of significant digits, then plus or minus one unit is allowed in the last digit. This idea here is to avoid favoring calculators which automatically round, or to place this burden on the contestant.

2. If the answer is given with more significant digits than the answer key requires, then the contestant's answer must round exactly to the answer key to receive the +3 points. If the last digit written is a 5, then the benefit of the doubt goes to the student, and we allow it to be rounded either up or down if either results in the correct answer according to the answer key.

3. If the answer is given with fewer significant digits than the answer key requires, then for +3 points to be awarded, the answer key must round exactly to the student's answer, provided the student gives at least two significant digits.

4. It would be possible for a significant-digit problem to have an answer like 540 (3SD). If this number were in the problem statement, it would be read as a 2SD number, but our notation permits us to make the trailing zero significant. In this case, answers of 539, 540, and 541 would be considered correct.

5. We never mix significant-digit problems with integer or dollar sign problems.

Absolute and Relative Error. The basic idea is simple enough: write a number of limited accuracy such that every digit is significant, i.e., so that it does not mislead the reader. Thus, if we write 3.456, the reader may assume that every digit, and in particular the last digit, really means something. In dealing with inexact numbers such as would result from measurement of a physical quantity, by writing 3.456 we are saying that this number is closer to the true but unknown value than 3.455 and 3.457. In mathematical terms, we would say that $3.4555<m_1<3.4565$, where m_1 is the quantity being measured. If we treat the (unknown) number as 3.456, then the largest absolute error we will make is E_1, and the largest relative error we will make is e_1, where

$$m_1 \approx 3.456 \pm E_1 = 3.456(1 \pm e_1)$$

Here, $E_1 \leq 0.0005$ and $e_1 \leq 0.0005/3.456 = 1.5x10^{-4}$. In this equation, the 3.456 is now considered an exact number of unlimited accuracy, all uncertainty being assumed by E_1 or e_1. Note that the maximum absolute error is half of a unit in the least digit of the number, and that the relative error is related to the number of significant digits written in the number. That is,

$$e_1 \approx 10^{-(\text{number of significant digits})} .$$

Assume that 0.00078 is another number relevant to this measurement. Using the same formulation as before, we write the measurement in this way.

$$m_2 \approx 0.00078 \pm E_2 = 0.00078(1 \pm e_2)$$

where $E_2 \leq 0.000005$ and $e_2 \leq 0.000005/0.00078 = 6.4x10^{-3}$. When a number is written with limited accuracy using E, we refer to the error as absolute. When written using e, we say the error is relative. Numbers written using E and e are written with absolute and relative accuracy, respectively.

Multiplication and Division. If our work requires the multiplication of two measured numbers m_1 and m_2, this may be accomplished by writing the numbers in forms with relative accuracy.

$$m_1 \text{ x } m_2 \approx (3.456)(1 \pm e_1)(0.00078)(1 \pm e_2) = 0.00269568(1 \pm e_1 \pm e_2 \pm e_1e_2)$$

Note that in multiplication the relative errors offer the more convenient expression. In the product, the leading number 0.00269568 is considered exact: all uncertainty still resides in the e's. In the UIL Calculator Applications Contest, we always use inexact numbers with two significant digits or greater. Thus the e's are always smaller than 0.1, usually considerably smaller. Because the e's are small numbers, we certainly can neglect the e_1e_2 term, and indeed, we can in this case neglect e_1 relative to e_2 because they differ by a factor of approximately ten. Thus the product is given approximately as follows.

$$m_1 \text{ x } m_2 \approx 0.00269568(1 \pm e_2) = 0.00269568 \pm 0.000017(\text{max error})$$

If we wish to drop the explicit statement of the errors and merely express the product with one number, we must make sure that every digit we write is significant. Noting that the uncertainty appears in the fifth place to right of the decimal, we must therefore avoid writing that place in our result, since it would be uncertain by greater than one–half a unit. Thus we must round up the 9 and write the result as shown.

$$m_1 \text{ x } m_2 \approx 3.456x0.00078 = 0.0027$$

We make the following observations. First, the product can be written to two significant digits. This is the lesser (least) of the significant digits of the two component numbers being multiplied. Second, in multiplication, the relative error of the product will in general be dominated by the component number having the least accuracy.

This follows from writing the number with relative accuracy which shows the strong dependence of relative error on the number of significant digits in the component numbers.

Third, if we had the product of more than two numbers, the higher order products of their relative errors would again be negligible relative to the errors themselves, which would add or subtract randomly in the sum. This sum of relative errors would be dominated by the largest relative error, which would be associated with the component number having the least number of significant digits.

Last, consider that we have a large number of components in a calculation involving multiplication and division, all having comparable accuracy (say, all having three significant digits). In this case it would be possible to have an accumulation of relative errors which might degrade the final accuracy (to two significant digits); however, this is unlikely because some of errors would add and some subtract. A law of probability suggests that total error would accumulate proportional to the square root of the number of terms involved. For example, if 10 terms were involved, all of comparable accuracy, we would expect the error to be about three times that of a single item. Hence if we require a factor of ten to degrade the accuracy by one significant digit, then a large number of terms, approximately 100 terms, would be required on the average to make likely such a degradation of accuracy.

Division of uncertain numbers develops along similar lines. This is shown by looking at the reciprocal of a number, for division can be considered as multiplication by the reciprocal. If an uncertain number is of the form $m_3(1 \pm e_3)$, then

$$\frac{1}{m_3\,(1 \pm e_3)} = \frac{1}{m_3}\left[\frac{(1 \mp e_3)}{(1 - e_3^2)}\right] \approx \frac{1}{m_3}\,(1 \mp e_3) \quad .$$

In writing the last form of this equation, we ignored e_3^2 compared to unity. The order of the + and – does not matter because e is of uncertain sign. We conclude that the reciprocal is of the same form and that relative errors will combine for division as for multiplication.

We can conclude that, in a calculation involving multiplication and division of a moderate number of component numbers, it would be reasonable to write the answer with a number of significant digits which is the least number of significant digits of the component numbers. This is the method of least significant digits.

Significant-Digit Example 1. The volume of the earth is 1.0831579x10^{12} km^3, and its mass is 5.979x10^{24} kg. What is the average density of the earth? (g/cm^3,SD)

This is a unit conversion problem with two numbers of limited accuracy. The first number has 8 significant digits and the second number has 4 significant digits. The average density A is the answer. It may be written directly from the information provided in the problem, but it needs to be written in proper units. As all mathematical operations are multiplication and division, the answer is written with the same number of significant digits as the least accurate number, 4SD.

$$A = \left[\frac{5.979\text{x}10^{24}\ \cancel{\text{kg}}}{1.0831579\text{x}10^{12}\ \cancel{\text{km}}^3}\right]\left\{\frac{1000\text{ g}}{1\ \cancel{\text{kg}}}\right\}\left\{\frac{1\ \cancel{\text{km}}}{1000\ \cancel{\text{m}}}\right\}^3\left\{\frac{1\ \cancel{\text{m}}}{100\text{ cm}}\right\}^3 = 5.520\ \ (4\text{SD})$$

Significant-Digit Example 2. The mass of the earth is 5.979x10^{24} kg. How many iron atoms does this mass represent if there are 55.847 g/mole of iron and 6.02486x10^{23} atoms/mole? (atoms,SD)

The answer is obtained by interpreting the information other than the mass of the earth m as unit conversion factors. As all the numbers are multiplied or divided, the answer has the same number of significant digits as the least accurate number involved in the calculation, 4SD (from 5.979x10^{24} kg).

$$m = 5.979\text{x}10^{24} \cancel{\text{kg}} \left\{\frac{1000 \cancel{\text{g}}}{1 \cancel{\text{kg}}}\right\} \left\{\frac{1 \cancel{\text{mole}}}{55.847 \cancel{\text{g}}}\right\} \left\{\frac{6.02486\text{x}10^{23} \text{ atoms}}{1 \cancel{\text{mole}}}\right\} = 6.450\text{x}10^{49} \text{ atoms (4SD)}$$

If you wrote this answer in the answer blank, or 6.451x10^{49} or 6.449x10^{49}, then you would receive +5 points. If you wrote 6.45x10^{49}, you would only receive +3 points, because your answer was written with only 3SD accuracy. 6.46x10^{49} would be counted incorrect, –2 points, because the correct answer does not round exactly to this answer. The same considerations are valid when the answer you write has more significant digits than the correct answer.

Significant-Digit Example 3. In 1949 Capt. James Gallagher was first to fly nonstop around the world, originating and ending in Fort Worth. The trip lasted 94 hr 1 min. If he flew along a constant latitude and Fort Worth is 32.8° north of the equator, what was his average velocity? The Earth's diameter is 7,926 miles. (mph,SD)

This is a rate problem. The distance d that Gallagher flew was the circumference of a circle whose radius R is 7,926 cos(32.8°)/2 mi = 3,331.165 mi (exact since none of these numbers were underlined in the problem statement). The distance then is 2πR = 20,930.330 mi (exact). The time needed to fly this distance is 94 hr 1 min (inexact). How do we determine the number of significant digits in a number when it is written in multiple units? This is common in time measurement and also in length measurement (e.g., 4 ft 6.3 in). We get the significant digits by writing the number in terms of the finest or smallest unit, in this case "min". The time t then is 94(60)+1 = 5,641 min. Now, we can look at this number and determine that there are 4SD. The velocity v comes from the rate equation. Since all numbers are multiplied and divided, the answer has the same number of SD's as the number with the least number of SD's, namely, 4SD.

$$v = \frac{d}{t} = \frac{20{,}930.330 \text{ mi}}{5{,}641 \cancel{\text{min}}} \left\{\frac{60 \cancel{\text{min}}}{1 \text{ hr}}\right\} = 222.6 \text{ mph (4SD)}$$

Addition and Subtraction. On the other hand, if we were required to add m_1 and m_2, we would have to work with the absolute accuracy of each, as shown by the calculation below.

$$m_1 + m_2 \approx (3.456 \pm E_1) + (0.00078 \pm E_2) = 3.45678 \pm E_1 \pm E_2$$

In the sum, E_1 would dominate and we can neglect E_2 in the sum.

$$m_1 + m_2 \approx 3.45678 \pm E_1 = 3.45678 \pm 0.0005$$

Because the maximum error appears in the fourth place to the right of the decimal place, we cannot write that place and retain significance of all numbers; hence, we must round to the third place to the right of the decimal place, writing the sum as 3.457. When we add inexact numbers, the number having the greatest absolute error dominates the accuracy of the result. Thus the answer should be rounded to the power of ten where the greatest absolute uncertainty of the component numbers occurs.

The same rule applies when a large number of numbers is involved. As before, if all the numbers are of comparable absolute accuracy, then errors can accumulate, but such accumulation will grow as the square root of the numbers contributing to the error in that place. In applying the addition rule, one should first add numbers, then round to the place of the greatest uncertainty. Errors would accumulate faster if the rounding were done before addition or subtraction.

When we subtract inexact numbers, the same rules apply. With subtraction of numbers of comparable magnitude, however, the results of the subtraction can be considerably smaller than the numbers being subtracted. Hence, the relative error can be larger than for either of the numbers. This can result in a reduction of the number of significant digits. For example, if we subtract 1.234 from 1.24462, the answer is 0.01062. The first number is accurate in the thousandths place, and so the answer must be stated only to that place, as 0.011. Here, by subtracting, we have dropped from 4 and 5 significant-digit accuracy (4SD and 5SD by our notation) to 2SD accuracy.

To conclude, for addition or subtraction, make the last significant digit of the result the same power of ten as the last significant digit of the component number of least absolute accuracy (or greatest absolute uncertainty). In such operations, the number of significant digits can change.

Significant-Digit Example 4. On January 1, 1988, the Berlin Zoo had 16,390 animals, including 4,720 fishes, 6,460 invertebrates, 480 amphibians, 2,910 birds, 330 reptiles and the remainder mammals. What percent of the animals were mammals? (%,SD)

The non-mammalian animals are numbered to tens place accuracy, remembering that trailing zeros of this type are not significant by our notation. The number of "other" animals O sums to 14,900, which has significance to the tens place by our addition rule, or 4SD. Here, one of the trailing zeros is significant and one is not. This can occur with numbers computed during a problem's solution, but it cannot happen in a problem statement unless we use scientific notation. The percentage of mammals comes from the formula given in two forms below, where T represents the total number of animals, 16,390 (4SD). The second form is preferred because it avoids repetition of inexact numbers (to be discussed shortly) and fewer keystrokes as well.

$$A = 100 \left[\frac{T - O}{T}\right] = 100 \left[1 - \frac{O}{T}\right] = 100 \left[1 - \frac{14{,}900}{16{,}390}\right] = 100 \; [1 - 0.9091] = 100 \; [0.0909]$$

$$A = 9.09\% \; (3SD)$$

The significant-digits part of the calculation begins with the division in the third formulation. Both numbers have 4SD accuracy, so the result, 0.9091, also has 4SD accuracy. When we subtract it from one (one is exact) in the next step, the addition/subtraction rule applies and we must write the result to the nearest ten-thousandth's place, 0.0909 (3SD). The subtraction has caused us to lose 1SD of accuracy, a common feature of subtraction of inexact numbers as described above. Multiplication by 100 (exact) produces our answer with 3SD accuracy.

If we had worked the problem using the first form of the equation, we would have obtained the same answer with the same number of significant digits. However, the significant-digits part of the solution may sometimes be incorrectly solved if an inexact number is repeated.

Significant-Digit Example 5. Oxygen exists in three isotopes: 99.759 % ^{16}O, and 0.2039 % ^{18}O. What is the percent occurrence of ^{17}O? (%,SD)

The two isotopes comprise 99.963% (5SD) of all oxygen. The percent occurrence of the remaining isotope is 100% minus this number, written to the thousandth's place, or 0.037% (2SD). Here the subtraction drops the accuracy from 5SD to 2SD.

Fractional Significant Digits. Despite the sound logic underlying the significant-digit approach, it is simply too crude to say that 11 and 99 both have two significant digits and hence are equally precise. We therefore will introduce a definition of significant digits which is compatible with, but refines considerably, the usual understanding of that term. We make this definition solely for the purpose of illuminating some of the limitations and difficulties of this method of dealing with inexact numbers. We have no ambition for introducing this definition into the actual Contest, making it a factor in the training of contestants or in the designing and conducting of tests. Taking up the ideas introduced earlier, we define the number of significant digits in a number N to equal SD(N), as

$$SD(N) = \mathrm{Log}_{10}\left[\frac{N}{E}\right]$$

where E is the maximum uncertainty in N. This is equivalent of the definition SD(N) = $-\text{Log}_{10}(e)$, where e is the maximum relative error (or uncertainty) in N; i.e., $N(l - e) \leq$ true value $\leq N(l + e)$.

Some examples: SD(11) = log(11/0.5) = 1.3SD, because 0.5 is the maximum uncertainty in 11. Similarly, SD(51) log(51/0.5) = 2.0SD, and SD(99) = log(99/0.5) = 2.3SD. For the numbers falling between 10 and 100, all of which we would normally call 2SD, the actual significance SD(N) falls between 1.3 and 2.3SD.

In general, E will be one-half a unit in the last significant digit of the number. We might write the significant digits of 0.0159 as SD(0.0159) = Log_{10}(0.0159/0.00005) = 2.5SD, and the significant-digit part of 6,655,100 would be SD(6,655,100) = Log_{10}(6,655,100/50) = 5.1SD. This shows that the SD function gives results in accord with our usual understanding, yet with a degree of refinement. Note that the significant digits is independent of the absolute value of the number: SD(5.1×10^n) = 2.0SD, independent of n.

The SD function allows us to examine with increased precision the conveyance of accuracy when we combine inexact (or uncertain) numbers. For example, let us multiply 11 by 51 according to the usual rules. The answer is 561, but we require 2SD, so we must give 560 as the answer. Let us now examine the operation with our refined definition. We gave above the SD's for 11 and 51 as 1.3 and 2.0, respectively. What is SD(11x51)? Because the maximum uncertainty in the component numbers is one half a unit in the last significant digit, the maximum uncertainty is E = 11.5x51.5 – 11x51 = 31.25. Thus SD(11x51)= Log_{10}(561/31.25) = 1.25SD, slightly less than the value of 1.3 for 11. This slight decrease in significant digits reflects the compounded error due to the uncertainty of both numbers. The result of the usual method, 560, has a SD(560) = 2.0, an overestimate of the true accuracy of the result by our definition, and an indication of the crudeness of the standard procedures.

If we add 11 and 51, we get 62 (2SD). The true SD of the sum is SD(11+51) = Log_{10}(62/1.0) = 1.8SD. Compared with the results of the usual rules, SD (62) = log(62/0.5) = 2.1SD, again an overstatement of the precision of the result.

To summarize, we might say that our more precise definition of significant digits confirms the validity but also discloses the weakness and crudeness of the standard methods for specifying and determining significant digits. Especially near the low end of each decade, as with 11, 1,100, 0.0011, etc., the number of significant digits is seriously overstated as 2SD.

On the other hand, the traditional method is simple and it guards us against gross error in dealing with inexact or uncertain numbers. The proverb about lighting a candle rather than cursing the darkness fits this situation: without introducing considerable complexity, we have no better method for specifying the precision of inexact numbers. Other methods have been developed for dealing with inaccurate numbers, which are generally superior to the method of significant digits, although at a cost of increased complexity. For this reason, we do not include them in the scope of the UIL Calculator Applications Contest.

Powers and Roots. Nonlinear operations such as powers, roots, and trigonometric expressions lead to another type of difficulty with significant-digit problems. Take for example the nth power of an inexact number.

$$[m_1(1 + e_1)]^n = m_1^n\left[1 + ne_1 + \frac{n(n-1)}{2!}e_1^2 + \ldots\right]$$

Here we have expanded the function with the binomial theorem. If we assume that e_1 is small, then the square and the higher order terms in e can be dropped and we are left with a relative error of ne_1. Here there occurs a linear increase in relative error, unlike the earlier case where error accumulated at worst according to a square root law. This linear increase can result in a change in the significant digits of the number, even if no other number is involved. An extreme example would be the tenth power, which would have one less significant digit than the number being raised to the tenth power.

To place this on a sound theoretical basis, we must generalize our definition of significant digits to include functions. Let us say we have a function f(m), where m is our inaccurate number having an absolute uncertainty of E. Then the uncertainty in f(m) would be $E_f = f(m + E) - f(m)$, and the relative uncertainty of f(m) would be $e_f = [f(m + E) - f(m)]/f(m)$. The significant digits of such a function could be written as follows.

$$SD[f(m)] = \text{Log}_{10}\left|\frac{f(m)}{E_f}\right| = \text{Log}_{10}\left|\frac{f(m)}{f(m+E) - f(m)}\right|$$

The vertical bars indicate absolute value, as required to avoid attempting the Log of a negative number. We can illustrate this with the square of an inexact number:

$$SD[(3.27)^2] = \text{Log}_{10}\left|\frac{3.27^2}{(3.275)^2 - 3.27^2}\right| = 2.5$$

Thus the significant digits of the square of 3.27, a 2.8SD number, is a 2.5SD number. It is easy to show that squaring will always lose 0.3SD, cubing will lose 0.5SD, and in general the nth power will lose $\text{Log}_{10}n$ SD.

On the brighter side, the taking of roots can lead to an increase in the precision of a number and a corresponding increase of significant digits. Indeed, the same law applies as for powers, with square roots gaining 0.3SD, cube roots gaining 0.5SD, etc. Check it out!

Significant-Digit Example 6. The handbook volume of the earth is 1.0831579x10^{12} km^3. A student estimated its mean radius to be 6,370.949 km. What is the percent error of the student's mean radius compared to the radius computed from the volume? (%,SD)

The radius computed from the volume R_v is calculated using the volume of a sphere.

$$V = \frac{4\pi}{3} R_v^3 \qquad \text{or} \qquad R_v = \sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{\frac{3\,(1.0831579\text{x}10^{12}\text{ km}^3)}{4\pi}} = 6{,}370.9039 \text{ km}$$

This radius has 8SD accuracy since the operations are multiplication, division and roots and the least significant-digit number (in fact, it's the only inexact number) has 8SD. We use the percent error formulation from the Problems Involving Functions section to get an answer.

$$\%\,Error = 100\left[\frac{Approximate\ Number}{Exact\ Number} - 1\right] = 100\left[\frac{6{,}370.9039\ km}{6{,}370.949\ km} - 1\right]$$

$$= 100\,[0.9999929 - 1] = 100\,(-0.0000071) = -0.00071\% \text{ (2SD)}$$

The ratio of radii has 7SD accuracy since division is involved and the least-significant-digit number involved in the calculation has 7SD. However, when we subtract 1 (exact) from the result, the result has an accuracy to the ten-millionth's place. This difference causes the significant digits to drop from 7SD to 2SD in the answer. As described below, we strongly discourage use of the percent formulas in the form 100(A -E)/(E), because it sometimes gives an incorrect SD answer even when the rules are correctly followed. This is elaborated below.

Trigonometric Functions. Rather that develop an elaborate theory about trigonometric functions, let us examine a typical case. Using the SD[f(m)] equation above, we compute SD[sin 35.62°] = 3.9, and thus the significant digits are preserved and even slightly increased. This is normal for sine and cosine functions, but with tangent and some of the other functions one can get a decrease in significant digits. Generally, the significant digits behave themselves nicely for trigonometric functions except at the extreme regions, such as cotangent of a small angle, tangent of an angle near 90°, etc., where one might expect trouble.

In summary, nonlinear functions lead to changes in the significant digits of inexact numbers. For the most common functions such as squares, square roots, and trigonometric functions, the changes are relatively small. For extreme cases like exponential functions, and other functions near singularities, one should be cautious and monitor significant digits using the SD[f(m)] relation. We plan to avoid the extreme cases when designing significant-digit problems for the Calculator Applications Contest. Hence, you can apply the simple rules without hesitation, provided calculations are sequenced so as to avoid repeating numbers.

Repeating Inexact Numbers in a Calculation. To pick a transparent but trivial example, let us say we have an operation involving two numbers, 512(3SD) and 13(2SD), and we sequence the calculation so as to multiply and divide by the same number.

$$512 = \frac{512(13)}{13}$$

On the left we have three-significant-digit accuracy, but by the usual rules, ignoring the cancellation of the identical number in numerator and denominator, we would specify 2SD on the right side of the equation, because two 2SD numbers are involved. The key point here is that the uncertainty of the 13 is allowed to dominate the final result, even though that uncertainty never affected the result due to cancellation. There is no real paradox here; just a warning that when the same number enters more than once into a calculation, the significant digits can be affected in perverse ways.

Here is an example not so obvious: What number is 1.1 percent less than 1201? There are two ways to make the calculation. We could write our answer as A by this computation.

$$A = 1201(1 - 0.011) = 1190 \text{ (3SD)}$$

The difference 1 – 0.011 is 0.989 (3SD). We multiply it by a 4SD number, 1201, and the result has 3SD by our rules. On the other hand, we can take 1.1% of 1201 and subtract this value from 1201.

$$A = 1201 - 1201\text{x}0.011 = 1201 - 13 = 1188 \text{ (4SD)}$$

Now, the product 1201x0.011 has 2SD accuracy due to the limited accuracy of 0.011 (2SD). We take the difference, and by our rules, the answer is accurate to the one's place, yielding an answer with 4SD! Both ways are correct mathematically, and both are correct by the usual rules of significant digits, but different answers result. A careful analysis with our more precise definition of SD's shows the 3SD answer to be correct. The second method exaggerates the accuracy. This occurs because the second calculation treats the two 1201's as different numbers having independent uncertainty. Again, the method with the repeated number leads to the wrong answer.

We have investigated additional examples and find it to be a good rule of thumb to sequence calculations so as to avoid repeating a number if possible. This is in general good practice, for it preserves accuracy and eliminates the effort of reentering a (rounded) number via the keyboard.

The significance of these factors for the Contest are as follows; we will design significant-digit problems with care and examine answers with these principles. You may be sure that your answer will be correct if repeated numbers are avoided, and if you follow the SD principles. We also will use few numbers near the bottom of the decade, such as 11, 102.5, etc., as these numbers tend to exacerbate difficulties. We would urge anyone making up significant problems for non-UIL tests to follow these precautions.

Significant-Digit Example 7. Sam observed that the population of Texas doubles about every 40 years. It was 3,896,542 in 1910. If it was 16,370,000 in 1985, what is the percent error in Sam's prediction? (%,SD)

This is a function problem, with combined exponential growth and percent error. Sam's prediction for Texas' population P is derived from given information in the form of a growth equation.

$$P = (3{,}896{,}542)\ 2^{(1985-1910)/40} = 14{,}292{,}580$$

This number with 7SD accuracy is the approximate number in the % error sense, with the inexact 16,370,000 (4SD) being the "exact" number. The result has 4 SD accuracy, as a result of the subtraction.

$$\%\,Error = 100\left[\frac{A}{E} - 1\right] = 100\left[\frac{14{,}292{,}580}{16{,}370{,}000} - 1\right] = 100[0.8731 - 1] = 100[-0.1269]$$

$$= -12.69\%\ (4SD)$$

The division retains 4SD accuracy, and the subtraction of 1 from 0.8731 also retains 4SD accuracy, not because we started with 4SD but rather because the difference between 0.8731 and 1 was large. The result of the subtraction must be written to the ten-thousandth's place since that is the place of least accuracy in the numbers contributing to the result. The answer has 4SD.

Suppose we had written the equation in the form which repeated the inexact number E. The solution would look like this.

$$\%\,Error = 100\left[\frac{A - E}{E}\right] = 100\left[\frac{14{,}292{,}580 - 16{,}370{,}000}{16{,}370{,}000}\right] = 100\left[\frac{-2{,}077{,}420}{16{,}370{,}000}\right] = 100[-0.127]$$

$$A = -12.7\%\ \ (3SD!)$$

The numerator containing the difference gives an intermediate answer of –2,077,420. However, the larger number in the difference, 16,370,000, is accurate only to the ten-thousand's place (4SD), so the difference is also accurate to this same place (e.g., –2,080,000, or 3SD). We find the final answer by dividing by a 4SD number and multiplying by 100 (exact). The answer by this approach should be written with 3SD. This is not correct by our rules because it involves unnecessary repeating of an inexact number. Also, we remind you that the SD rounding should only take place for the final answer, not for any intermediate results.

Problem-Solving Summary. Significant-digit problems are denoted by underlined numbers and "SD" in the answer blank. Trailing zeros in an underlined number are by our definition not significant if they do not cross the decimal point. The number, 57,500, would be interpreted to have 3SD, but 57,500.0 would have 6SD accuracy. Answers to significant-digit problems must have at least 2SD accuracy. Any conversion factors you provide are considered exact even if they are intrinsically approximations. The conversion factor, 7.481 gal/ft^3, is considered exact for purposes of solving significant-digit problems if it is not underlined in the problem statement, even though in reality it is written to 4SD accuracy.

For all operations except addition and subtraction (multiplication, division, powers, roots, trigonometric functions, etc.), you do what makes sense. The result has the same number of significant digits as the number with fewest significant digits. In the example below, all numbers are inexact (i.e., underlined in the problem statement) except 5.5. The numbers are 5.5 (exact), 45.766 (5SD), 0.00312 (3SD), 87.355 (5SD), –55.48 (4SD) and 44.59 (4SD). If you have trouble with how the significant digits were obtained, read the section on how to write answers in Chapter 2D.

$$\frac{(5.5)(45.766)(0.00312)(87.355)^2}{(-55.48)\sqrt{44.59}} = -16.2 \text{ (3SD)}$$

The result has 3SD since that's the lowest value of significant digits in the inexact numbers involved in the calculation. (Remember that the 5.5 is exact and doesn't count in the SD part of the calculation.)

For addition and subtraction, don't worry about how many SD's the numbers have. The key is to round the answer to fit the absolute accuracy of the least accurate number in the calculation. Suppose we have two inexact numbers 7,931.266 (7SD) and 7,930.924477 (10SD). Forget about the fact that one number has 7SD and the other one has 8SD. What is important is that the first number is accurate to the thousandth's place and the second one is more accurate, being written to the millionth's place. The accuracy of the least accurate number (the first one) is to the thousandth's place, and the result of addition or subtraction of these numbers is to the same level, the thousandth's place. The sum of the numbers is 15,862.190 (8SD). We picked up one SD because the summation pushed the result from a thousand's number to a ten thousand's number.

If we subtract these numbers, the result is still accurate to the thousandth's place. The answer is 0.342 (3SD). This illustrates a problem that occurs often in experimental measurements. When subtracting inexact numbers, especially numbers that are almost equal to each other, the result has much reduced accuracy relative to the starting numbers. For example, suppose we wanted to measure the width of this Contest Manual page. If you put a ruler across the page, perhaps you could measure it to be 8.5 inches with an absolute accuracy of 0.1 inch, or 2SD. Now imagine that we measure the page width from Austin, Texas, where we are. We do this by running a very long ruler from us to the page. It might be as much as 500 miles from where we are to where you are, assuming you're in Texas, too, so our ruler might have to be accurate to 0.1 inch in 500 miles to reach you and to measure the page with the same accuracy as your 12-inch ruler. In other words, since 500 miles is 31,680,000 inches, our ruler will have to measure a number on the order of 31,680,000.0 inches (9SD) in order to measure the page width to 2SD accuracy. Our potential values might be 14,739,877.2 inches (9SD) to the left side of the page, and 14,739,868.7 in (9SD) to the right side. The difference is the page width, accurate to the nearest tenth's place, or 8.5 in (2SD).

The last rule for significant-digit problem solving is never to repeat a number in the calculation of SD's for the answer. The most common place where this occurs is in percent-type function problems. That's why we recommend writing percent error problems (for example) as

$$\%\text{Error} = 100\left[\frac{A}{E} - 1\right] \quad \text{rather than as} \quad \%\text{Error} = 100\left[\frac{A - E}{E}\right] .$$

Clearly, either will give the same numerical answer. It's when you try to figure out how many significant digits of the answer to write in the answer blank that not repeating the same inexact number becomes potentially important.

You should work significant-digit problems without any intermediate rounding. Only the final answer should be rounded.

<u>Significant-Digit Example 8.</u> Light travels at <u>186,000</u> miles per second and sound travels at <u>1,096</u> feet per second. If thunder is heard 8.5 seconds after lightning is seen, how far away is the observer from the lightning? (mi,SD)

This is a rate problem. Light travels so fast relative to sound that we can assume that the light reaches us instantaneously and only worry about the sound velocity. The validity of this neglect is illustrated in this problem along with some insights into significant-digit computation. The observer's distance from the lightning d is the unknown. Light travels this distance in time t_l, and sound travels the same distance in t_s, according to the rate equation.

$$t_l = \frac{d}{v_l} = \frac{d\,sec}{186{,}000mi} \text{ and } t_s = \frac{d}{v_s} = \frac{d\,sec}{1{,}096ft}$$

The time interval, 8.5 sec, is the difference in these times, from which d may be solved.

$$8.5\,sec = t_s - t_l = \frac{d\,sec}{1{,}096\,ft} - \frac{d\,sec}{186{,}000\,mi} = (d\,sec)\left[\frac{1}{1{,}096\,ft}\left\{\frac{5{,}280\,ft}{mi}\right\} - \frac{1}{186{,}000\,mi}\right]$$

$$d = \frac{8.5\,sec}{\frac{sec}{mi}\left[\frac{5{,}280}{1{,}096} - \frac{1}{186{,}000}\right]} = \frac{8.5\,mi}{4.818 - 0.00000538} = \frac{8.5\,mi}{4.818} = 1.764\,mi\ (4SD)$$

The significant-digits part of the solution picks up in the last line. The ratio 5,280/1096 is given with 4SD accuracy (don't round numbers on your calculator until you get the final answer!), and the reciprocal of 186,000 is given with 3SD. The role of light is completely washed out when the subtraction is done, because the least accurate place in the sound term, the thousandth's place, is less accurate than the first significant digit in the light term, the millionth's place. The result of the subtraction is given to the thousandth's place, noting that the accuracy or inaccuracy with which the speed of light was measured does not affect our answer. The answer then has 4SD accuracy since the 8.5 seconds was considered in this problem to be exact. That is, it was not underlined in the problem statement.

Chapter 5

Geometry Problems

A. Introduction

Like the stated problems on the Calculator Applications Contest, the 14 geometry problems may be categorized into specific types. Unlike most stated problems though, the geometry problems are also classified by location on each test. The type and location are given with brief explanation in the UIL Calculator Applications Practice Manual for Stated and Geometry Problems. The table shows this relationship in summary form. On the

Table: Classification of Geometry Problems on UIL Calculator Applications Contests

Problem Number	Problem Type
9, 10	Simple one-step calculation on a simple plane figure: circle, rectangle, trapezoid, triangle or a degenerative form (square, parallelogram, rhombus, semicircle, etc.)
19, 20	One-step solutions of right triangles
29, 30	One-step solid geometry problems
39	Inscribed and circumscribed circles
40	Law of Sines, Law of Cosines
49, 50	Solid geometry problems. Forms may be combined to create more complex figures
59	Calculus problem
60	Difficult plane geometry problem
64, 65	Generally complicated, multiple-step plane geometry problems

following page are sample geometry problems from each problem type. The unknown on geometry problems varies, but may in general be a periphery, side dimension or other linear dimension, angle, area or volume. All answers on geometry problems are to be written with the standard, three-significant-digit accuracy in either fixed or scientific notation.

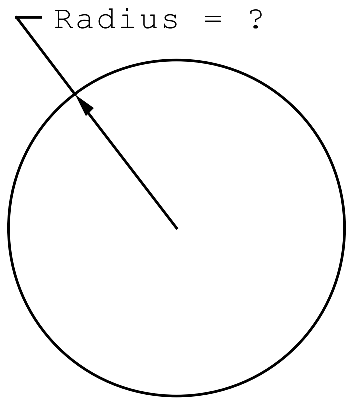
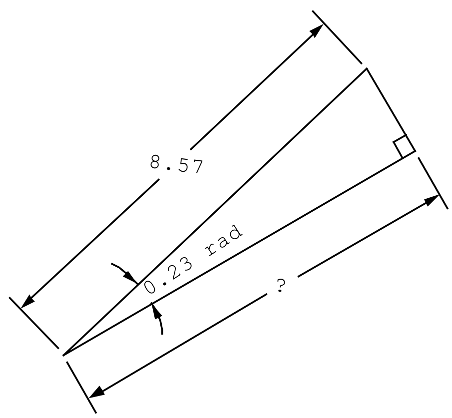
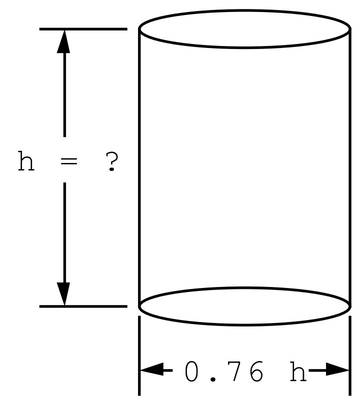
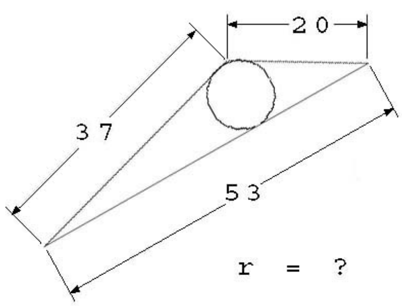
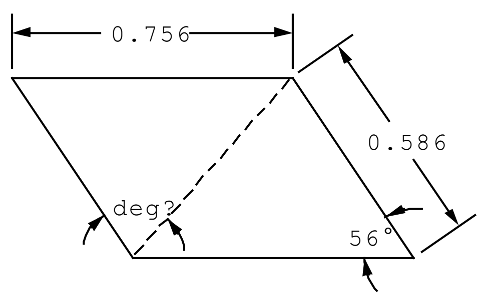
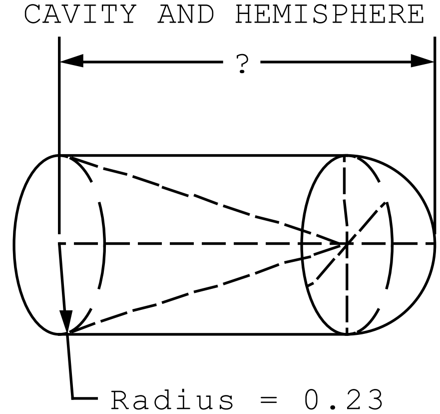
Students who have already taken courses in geometry, trigonometry and calculus have an obvious advantage over those students who are currently in lower high school grades. However, with some coaching and study, it is not impossible for freshmen and sophomores to work these problems.

B. Introduction to Trigonometry

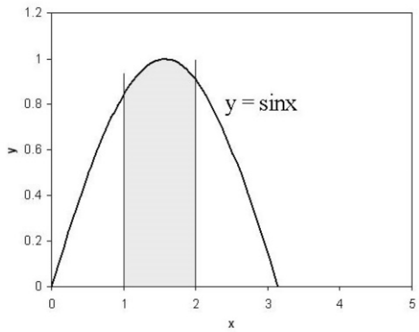
Trigonometry is the study of triangles, especially the relationships between their angles and side dimensions. We do not intend to include an exhaustive discourse on all aspects of trigonometry (or "trig" as it's often called), but rather to provide you with the basics of trigonometry needed for the Calculator Applications Contest. We assume you know about angular measure in revolutions, degrees, and radians. If not, you may want to refer to the section in Chapter 4Ei dealing with Problems Involving Rates where we define angular measure in the context of rotational motion.

Some definitions are in order. First, an "acute" angle is an angle whose measure is between 0 and 90° or 0 and π/2 radians, and an "obtuse" angle is one whose measure falls between 90° and 180°, or π/2 and π radians.

Figure 1. Geometry Problem Examples

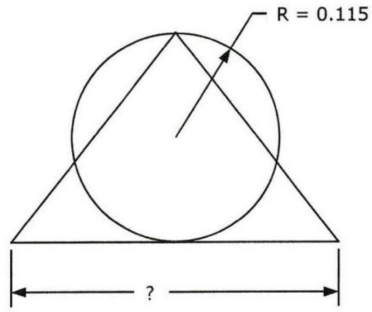
Page 1, Problems 9, 10 CIRCLE	Page 2, Problems 19, 20 RIGHT TRIANGLE	Page 3, Problems 29, 30 CYLINDER
 Area = 0.0295		 Volume = 948
Page 4, Problem 39 SCALENE TRIANGLE 	Page 4, Problem 40 PARALLELOGRAM 	Page 5, Problems 49, 50 CYLINDER WITH CONICAL CAVITY AND HEMISPHERE  Volume = 0.134

Page 6, Problem 59



Shaded Area = ?

Page 6, Problem 60
Page 7, Problems 64, 65
CIRCLE AND ISOSCELES TRIANGLE



Area(Circle) = Area(Triangle)

"Complementary" angles are two angles whose sum equals 90°. "Supplementary" angles are two angles whose sum equals 180°.

Let's look at what makes up a triangle. Figure 2a shows one labeled in a conventional format. The side dimensions are given by lower case letters. The angles are given by capital letters. The side dimension "opposite" an angle shares the same letter. That is, Sides b and c are the connecting lines making up Angle A, and so on. This is not always the case, but unless there is a reason not to, the length of Side a (or just "a") is smaller than that of Side b (or just "b"), and the length of Side c (or just "c") is the largest.

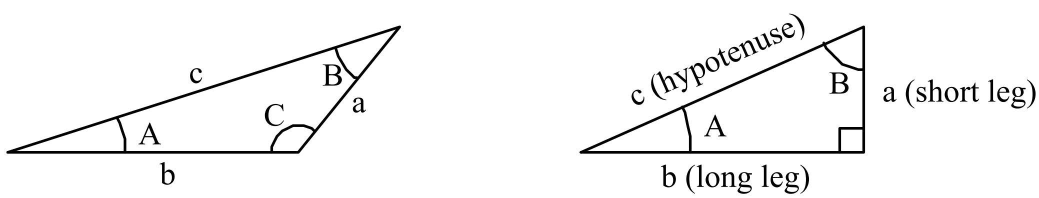


Figure 2

When the largest angle equals 90° or $\pi/2$ radians, a "right" angle, we have a triangle like the one shown in Figure 2b. This type of triangle is called a "right triangle", and the right angle is usually given by a small box symbol without any numbers. The side dimensions take on special names. Sides a and b are called the "legs" of the right triangle and are distinguished from one another by reference to "the short leg" or "the long leg". The longest side, Side c opposite the right angle, is called the "hypotenuse". All plane trigonometric functions relate the angles and side dimensions of right triangles.

Figure 3 was made by constructing a series of right triangles with a constant Side b. Can you see what changes? Side a increases in length from left to right. However, note that the measure of Angle A also increases, and since one end of the hypotenuse is stuck to an end of Side a, the length of the hypotenuse also increases. We see here that (for Side b constant) that the length of Side a and the measure of Angle A are related. Increasing Side a makes Angle A increase.

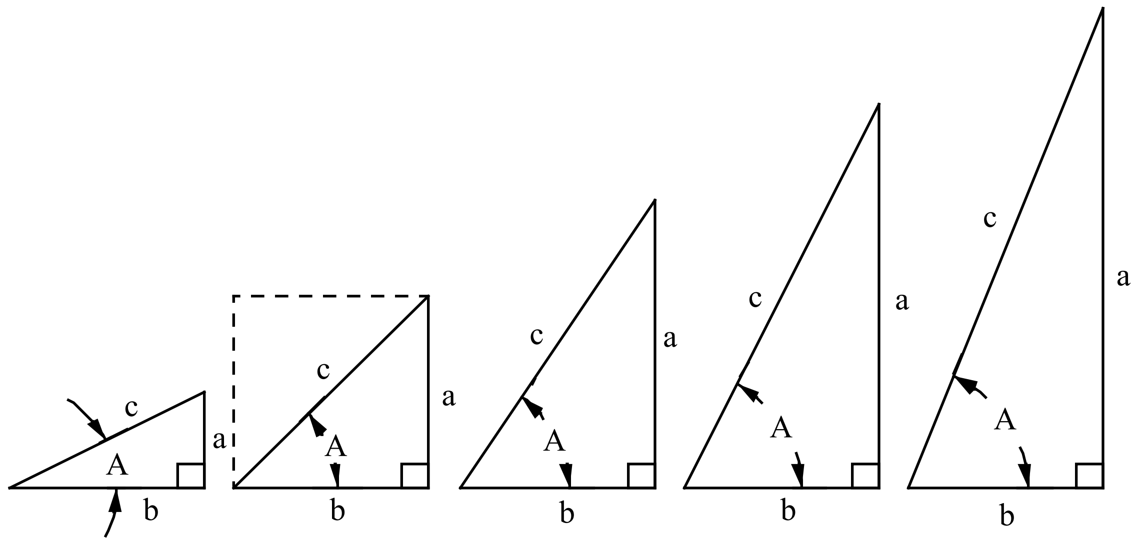


Figure 3

Thinking about the extremes of Side a, we can observe specific values of Angle A. Suppose the length of Side a were zero. Then Angle A would also measure zero. This funny-looking, zero-angle triangle would look like a horizontal line of length b, and we would place it as the left-most triangle in Figure 3. What would the right-most triangle look like? If we made Side a larger and larger, eventually Side a and the hypotenuse c would look almost parallel. Angle A would approach a right angle, 90° or $\pi/2$ radians. Another way to say this is, when $a \rightarrow \infty$, $A \rightarrow 90°$ or $\pi/2$ radians. Look at the second triangle from the left in Figure 3. Sides a and b have equal length here, and the measure of Angle A is 45° or $\pi/4$ radians. The dotted lines form a square with the legs of this triangle to help you visualize this observation. We are now ready to define our first trigonometric function, tangent A. The tangent of Angle A, denoted "tan(A)" or "tanA", is the ratio of the length of Side a to that of Side b:

$$\text{tangent}(A) = \tan(A) = \tan A = \frac{a}{b} \quad . \tag{1}$$

Look at your calculator. First, you need to set the angle notation to either degree or radian measure. Look in the instruction manual or check with someone to see how to do this. However, most scientific calculators default in degree notation, so you may be able to keep reading on even though you might not know how to change angle mode on your calculator. Since angle measure in degrees is more popular generally than radian measure, we will deal only with degree measure for the remainder of this chapter, acknowledging that on the Calculator Applications Contest, we work with trigonometric functions both in degree and radian notation.

You should have a "tan" button on your calculator somewhere. Punch in zero and hit the "tan" button. You should have taken the tangent of zero degrees, and from our previous discussion, the value of a is zero, so a/b is zero. Since this is the tangent of A by the above equation, your calculator should show the result, zero. When Angle A measures 90°, the tangent of A is infinite. Punch in "90" and hit the "tan" button. Your calculator should display its version of infinity or an error message saying the number is too big for your calculator to handle. If your display shows a number like –1.995200412, then your calculator is set in radian mode and you just took the tangent of 90 radians. Enter "45" into your calculator and punch the "tan" button. You should see unity as the result, reflecting the fact that when Angle A measures 45°, then a and b are equal, making their ratio 1. Play around with other angles, taking the tangent of each. The respective angles in the Figure 3 triangles measure approximately 27°, 45°, 56°, 63° and 68°. The respective length ratios (a/b) are 0.5, 1, 1.5, 2 and 2.5.

Now that you know how to take the tangent of an angle to get a ratio of leg lengths, how do you do the opposite? That is, what is the measure of Angle A when the ratio a/b equals 0.76? You can guess angles and punch the "tan" key until you get close to 0.76, but there's a much better way. We can write the above equation by solving for the angular measure A. Doing so gives us a new trigonometric function which has several names. They are the "arctangent" or "inverse tangent" of a number. The symbol is confusing at first. Generally, it looks like "Arctan(x)" or "Tan^{-1}(x)" where x is a/b:

$$\text{Arctan}\left(\frac{a}{b}\right) = \text{Tan}^{-1}\left(\frac{a}{b}\right) = A \quad . \tag{2}$$

Learn how to take the inverse tangent of 0.76 on your calculator. Calculators with shift keys usually place the inverse tangent as a "shift-tangent" operation. The answer is 37.235°.

It's easy to confuse this arctangent notation with reciprocal of a tangent or the tangent of the reciprocal of a number. Both are incorrect in that they do not equal the arctangent of a number. That is,

$$\text{Arctan}\left(\frac{a}{b}\right) = \text{Tan}^{-1}\left(\frac{a}{b}\right) \neq \left[\tan\left(\frac{a}{b}\right)\right]^{-1} \neq \tan\left[\left(\frac{a}{b}\right)^{-1}\right] \quad .$$

Another problem with the inverse tangent is that it is not a single-valued function. That means that the arctangent of a number has many answers. We saw earlier that the arctan(0.76) was 37.235°. This means that the tangent of 37.235° equals 0.76. What is the tangent of 217.235°? Yikes! It's also equal to 0.76, so the arctan(0.76) is also 217.235°! It turns out for the tangent and arctangent functions that we can add or subtract 180° as many times as we want to the angle A and the equality holds. Try this on your calculator. This "multi-valued-ness" doesn't matter when we take tangents, because we choose the angle to input. The problem arises when we use the inverse tangent function because the answer is in fact a set of angles all ±180n° apart where n is a positive integer. By calculator convention, your calculator gives you only one value, generally the value between –90° and +90°. If the angle you seek does not fall in this range, then <u>you</u> have to add or subtract 180° enough times to get within the range dictated by the problem. Fortunately, most of the time the angle we seek falls in the ±90° range. Suppose you wanted to solve this stated problem: "The tangent of what angle nearest 270° equals 2.61?" Letting the measure of the angle equal A, we can write

$$\tan(A) = 2.61 \quad \text{or} \quad A = \text{Tan}^{-1}(2.61)$$

Punching the numbers on the calculator produces A = 69.0°, but we can get closer to 270° by adding 180° to obtain A = 69.0 + 180° = 249°. This measure falls within 21° of 270°, so it is the answer we seek.

There are two other trigonometric functions similar to the tangent function. Looking back at Figure 2b, we see that the tangent of A is the ratio of a to b. We can define the "cosine" of Angle A as the ratio of b to c. Written mathematically, it appears as

$$\text{cosine}(A) = \cos(A) = \cos A = \frac{b}{c} \quad . \tag{3}$$

The cosine of an angle relates the angle to the ratio of the lengths of the two lines forming it. The other trigonometric function is the "sine" of Angle A, written

$$\text{sine}(A) = \sin(A) = \sin A = \frac{a}{c} \quad . \tag{4}$$

The tangent of an angle varies from $-\infty$ to $+\infty$. The sine and cosine of an angle span between -1 and $+1$. Find the "cos" and "sin" buttons on your calculator. Find cosA and sinA for the triangles in Figure 3. Do they increase or decrease as A increases?

In the right triangle of Figure 2b, we have three angles: 90°, A and B. Since the sum of the interior angles of a triangle is 180°, A and B sum to 90°. You can show that a/c is both sinA and cosB. Indeed, this leads us to the first relationship between trigonometric functions, namely

$$\sin A = \cos(90^\circ - A) \quad \text{and} \quad \cos B = \sin(90^\circ - B) \ . \tag{5}$$

Another relationship follows involving all three trigonometric functions.

$$\tan A = \frac{a}{b} = \frac{a/c}{b/c} = \frac{\sin A}{\cos A} \tag{6}$$

From the Pythagorean Theorem, we know that $a^2 + b^2 = c^2$. Dividing both sides by c^2, we obtain

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \quad \text{or} \quad (\sin A)^2 + (\cos A)^2 = 1 \tag{7}$$

When raising trigonometric numbers to a positive integer power, often the number is inserted between the function name and the angle argument of the function. Rewriting the last equation in this format, we obtain

$$\sin^2 A + \cos^2 A = 1 \ . \tag{8}$$

The trigonometric functions cosine and sine also have inverse forms like the tangent. They are

$$\arccos\left(\frac{b}{c}\right) = \mathrm{Cos}^{-1}\left(\frac{b}{c}\right) = A \quad \text{and} \quad \arcsin\left(\frac{a}{c}\right) = \mathrm{Sin}^{-1}\left(\frac{a}{c}\right) = A \tag{9}$$

These functions are also multi-valued, but with answers spaced twice as far apart as they were for the arctangent. That is, the inverse sine or cosine of a number equals some angle A as well as all angles ±360n° removed from A (n is any positive integer). Your calculator does not give Arcsin and Arccos over a range of 360° though. The reason is that two identities reduce the range. These identities are

$$\sin(180^\circ + x) = -\sin x \quad \text{and} \quad \cos(180^\circ + x) = -\cos x \tag{10}$$

You might try these out on your calculator with some angles. Arcsin and Arccos span only 180°, like the tangent function. As if matters weren't confusing enough, the span is different. Arcsin spans between –90° and +90°, like the inverse tangent function. The Arccos function spans between 0° and 180° though! Again, try this by taking the Arcsin and Arccos of numbers between –1 and 1 to locate their span on your calculator.

Knowing the trigonometric functions, tangent, sine and cosine, we can solve a right triangle completely (find the values of the two acute angles, the two legs and the hypotenuse) if we know any one of the lengths and either of the angles. Alternatively, we can solve it completely knowing only two of the three lengths and whether the lengths are legs and/or the hypotenuse. We can also solve for the area K of a right triangle since the area of a right triangle may be given in terms of the legs:

$$K = \frac{1}{2} ab \ . \tag{11}$$

C. Scalene Triangles (Laws of Sines and Cosines)

A triangle is called a scalene triangle when none of the three side dimensions, a, b and c, are equal and the triangle is not a right triangle, Figure 4. It is the most general triangle devisable. If two of the three sides are equal in length, we call the triangle an isosceles triangle. And when all three sides are equal in length, we have an equilateral triangle. Equally constrained as an equilateral triangle is a right isosceles triangle, often referred to as a "45° triangle". As the name implies, it is a right triangle with the two legs equal in length. The second triangle from the left in Figure 3 is a right isosceles triangle. All of the equations presented in this section are valid for any of these triangles, but you should know that most equations reduce to much simpler forms depending on the constraint of the triangle. For all triangles, the sum of the measures of the three angles always equals 180°.

$$A + B + C = 180° \qquad (12)$$

Every triangle has three altitudes. An altitude is a line segment that starts at one of the triangle's corners and projects to be orthogonal (form a 90° angle) with the opposite side or its projection. Figure 4 shows two scalene triangles and the three altitudes as dotted lines. The directions of two side dimensions of the second triangle were extended to define the respective altitudes. We label altitudes by the letter h with a subscript denoting the triangle leg which the altitude strikes orthogonally.

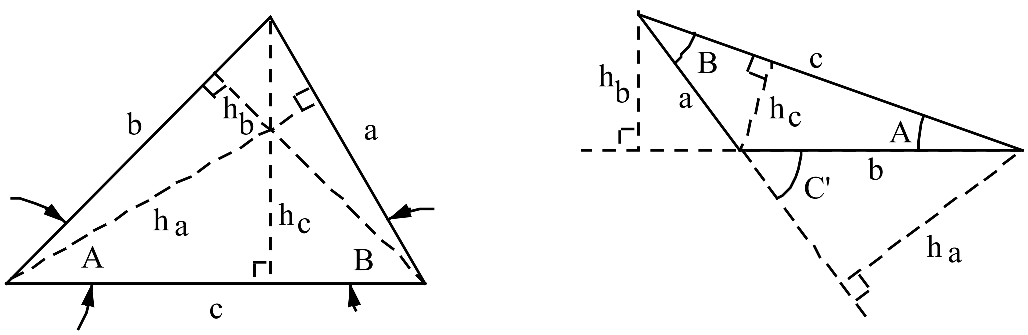


Figure 4

The magnitude of the altitudes can be obtained using trigonometry. Look first at h_c of the left scalene triangle in the figure. It and Side b form the leg and hypotenuse of a right triangle. The opposite angle is A, so $h_c = b \sin A$. This altitude also forms a right triangle with Side a as the hypotenuse. From this triangle, we obtain $h_c = a \sin B$. The other altitudes form right triangles with c as the hypotenuse. Gathering all these relations for the altitudes, we may write

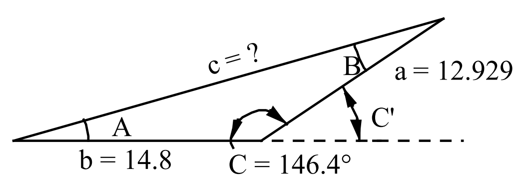


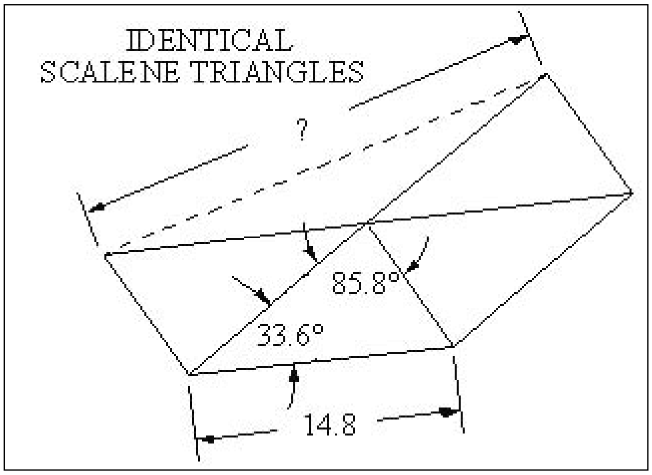
Figure 5

$$h_c = b \sin A = a \sin B \qquad h_a = c \sin B = b \sin C \qquad h_b = c \sin A = a \sin C \qquad (13)$$

From the altitude equations comes a very important equation called the "Law of Sines". From the second and third forms of each equation above, we can divide by a length dimension and obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{\sin C'}{c} \quad \text{(Law of Sines)} \quad . \qquad (14)$$

The last term is new and will be discussed a little later. Angle C' is shown on Figure 4 as the supplement of Angle C. The Law of Sines is used to relate unknown lengths or angles in terms of the other variable. We use it to



completely solve for any unknown in a scalene triangle when one side and two angles are known, or when two sides and the nonadjacent angle are known.

Geometry Example 1. See the figure showing identical scalene triangles.The four scalene triangles are identical or "congruent". This means they share the same lengths and angles. Two angles and one side are given. The third angle is 180°–85.8°–33.6° = 60.6°. The two remaining side dimensions of the scalene triangle come from the Law of Sines.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{or} \quad a = \frac{\sin A}{\sin C}\, c = \frac{\sin 33.6°}{\sin 85.8°}(14.8) = 8.212$$

$$\frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad b = \frac{\sin B}{\sin C}\, c = \frac{\sin 60.6°}{\sin 85.8°}(14.8) = 12.929$$

The dotted line answer we seek is the leg of a scalene triangle different from the others. It is shown separately as Figure 5. The other legs are 14.8 (left bottom leg) and 12.929 (right bottom leg) from the fact that the other scalene triangles are identical and have equal legs. The obtuse angle is obtained knowing that one revolution is 360°. This unknown angle, Angle C in the triangle sums clockwise around its apex with 33.6°, 60.6°, 85.8° and another 33.6° to equal one revolution or 360°. From this, C = 360°–[2(33.6°)+60.6°+85.8°] = 146.4°. This poses a problem. The Law of Sines does not help us when two sides and the adjacent angle are known. The reason is that the equations always give us two unknown angles or two unknown sides. We use this dilemma as an opportunity to present another important relationship between sides and angles in a scalene triangle. It is the equivalent of the Pythagorean Theorem for scalene triangles, and it is called the "Law of Cosines".

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{(Law of Cosines)} \tag{15}$$

The Law of Cosines differs from the Pythagorean Theorem in the last term. (You can show that this relation reduces to the Pythagorean Theorem when Angle C is a right angle.) The Law of Cosines is useful in cases like the present one, and also when all three side dimensions are known but no angles (See Geometry Example Problem 2 in Chapter 4E). Then, the Law of Cosines may be solved for the measure of Angle C. For our problem, we solve for c.

$$c = \sqrt{a^2 + b^2 - 2ab\cos C} =$$

$$\sqrt{12.929^2 + 14.8^2 - 2(12.929)(14.8)\cos 146.4°}$$

$$c = 26.6$$

You need to be careful when using the Law of Sines with obtuse angles. Remember that the arcsin button on your calculator gives an angle between –90° and +90°, so the Law of Sines gives the "wrong" answer when solving for an obtuse angle. Look at Figure 5. From the Law of Sines, we have calculated Angle A to equal 15.633°. Suppose we had a triangle like this one, and we knew only that A = 15.633°, a = 12.929 and c = 26.551. We want to know C. No problem. We use the Law of Sines, solved for C.

$$C = \text{Arcsin}\left(\frac{c}{a}\sin A\right) = \text{Arcsin}\left(\frac{26.552}{12.929}\sin 15.633°\right) = 33.6° \,!$$

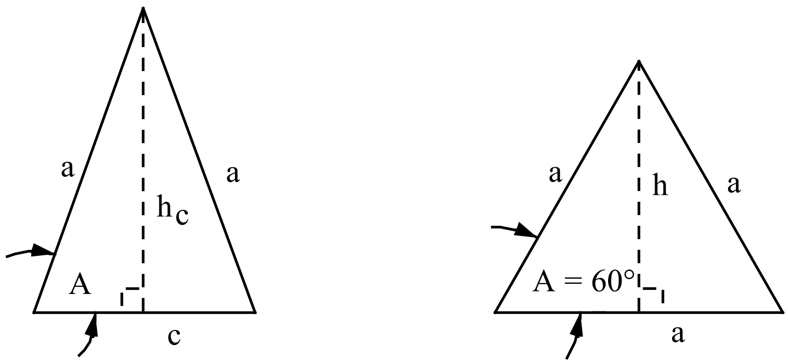


Figure 6 - Isosceles and Equilateral Triangle

This is clearly incorrect since the angle whose measure we seek is obtuse, not acute. The problem is that the Arcsin function on our calculator will not give us an obtuse angle for the Arcsin because the range of answers spans only between –90° and +90°. From Equation 10, we know that the sin(180°+x) = –sinx. Further, you can establish that –sinx = sin(–x) by trying some values on your calculator. From these we get a new trigonometric identity:

$$\sin(180°-x) = \sin(x)$$

That is, the sine of an angle equals the sine of its supplement. Looking at Figure 4b or Figure 5, the supplementary angle to C is given as C'. This is in fact the angle we have solved for. To obtain the associated obtuse angle, we simply subtract our answer from 180° to get 146.4°. To summarize, be careful when using the Law of Sines to solve for unknown angles. Always ask yourself, "Is the angle obtuse or acute"? If it's obtuse, make sure you adjust the "answer" as described.

The Law of Sines and the Law of Cosines provide the means to solve for any unknown in a scalene triangle, given any three of the six parameters: a, b, c, A, B, and C. The only constraint is that at least one dimension (lower-case letter) must be given.

The area of a scalene triangle is half the product of any leg times its associated altitude.

$$K = \frac{1}{2}\, a h_a = \frac{1}{2}\, b h_b = \frac{1}{2}\, c h_c \tag{16}$$

Substituting earlier equations for the altitudes, we obtain

$$K = \frac{1}{2}\, ab \sin C = \frac{1}{2}\, bc \sin A = \frac{1}{2}\, ac \sin B \quad . \tag{17}$$

Each form contains one of the three letters. As shown explicitly here and in the Law of Sines, there is no mathematical distinction between a, b and c or their respective angles. The same holds for the Law of Cosines, which may be written in the alternative forms:

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \text{or} \quad b^2 = c^2 + a^2 - 2ca \cos B \tag{18}$$

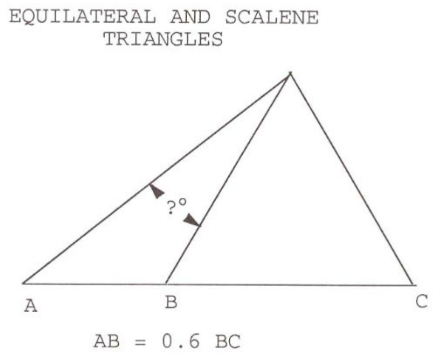
D. Isosceles and Equilateral Triangles

When two sides of a triangle are equal in length, the triangle is called an isosceles triangle. Often, by convention, the two, equal-length legs are given as a, with the unique leg equal to c, as shown in Figure 6. From the Law of Sines, A = B as well. The altitude h_c equals (a sinA). From the Pythagorean Theorem,

$$\left(\frac{c}{2}\right)^2 = h_c^2 + a^2 \quad \text{or} \quad c^2 = 4(a^2\sin^2 A + a^2) = 4a^2(1 + \sin^2 A)$$

This is a simpler substitute for the Law of Cosines. The area of an isosceles triangle K is

$$K = \frac{1}{2}\, ch = \frac{1}{2}\, ca \sin A = \frac{1}{2}\, c\left(\frac{c/2}{\cos A}\right) \sin A = \frac{1}{4}\, c^2\left(\frac{\sin A}{\cos A}\right) = \frac{1}{4}\, c^2 \tan A$$



An equilateral triangle has all sides equal. The internal angles are also equal from the Law of Sines, and since they sum to 180°, each angle equals 180°/3 or 60°. The altitude of an equilateral triangle equals (a sin60°), but the sin60° = $\sqrt{3}$ /2. Using this to form the area K, we get

$$h = \frac{\sqrt{3}}{2}\ a \quad \text{and} \quad K = \frac{1}{2}\ ah = \frac{1}{2}\ a\left(\frac{\sqrt{3}}{2}\ a\right)$$

$$K = \frac{\sqrt{3}}{4}\ a^2$$

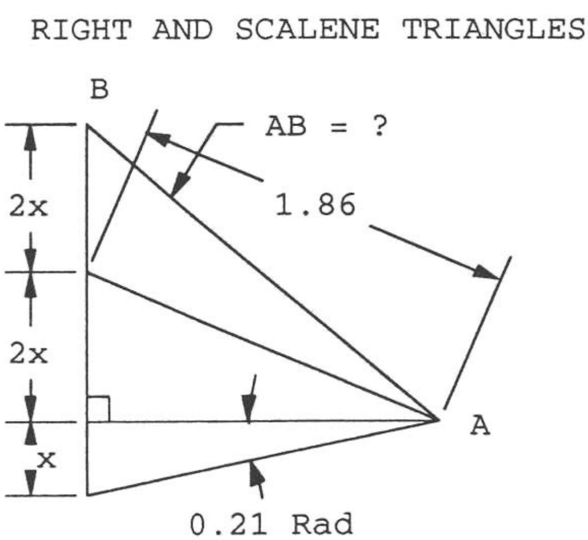
<u>Geometry Example 2:</u> See the figure on the next page showing an equilateral and scalene triangle.

This problem requires use of the Law of Cosines and the Law of Sines for its solution. The scalene triangle has a horizontal leg equal to 0.6 BC. The scalene triangle shares another leg with the equilateral triangle, so it equals BC, since all sides of the equilateral triangle are equal. The obtuse Angle B equals 120° since the acute angle in the equilateral triangle is 60° and is its supplement. We can write the Law of Cosines and solve for the unknown third leg of the scalene triangle.

$$c = \sqrt{a^2 + b^2 - 2ab\cos C}$$

$$c = \sqrt{(0.6\text{BC})^2 + (\text{BC})^2 - 2(0.6\text{BC})(\text{BC})\cos 120°}$$

$$c = 1.400\ \text{BC}$$



Now, from the Law of Sines, we can solve for the unknown angle.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin A}{0.6\ \text{BC}} = \frac{\sin 120°}{1.400\text{BC}}$$

$$\sin A = \frac{0.6}{1.400}\ \sin 120° = 0.371 \quad \text{or} \quad A = \text{Arcsin}(0.371)$$

$$= 21.8°$$

We have to ascertain whether the angle we seek is acute or obtuse. In this case, we want the acute angle, so the value we have calculated is the correct answer.

<u>Geometry Example 3:</u> Right and scalene triangles (below).

This problem may be worked several different ways. Generally, it is better (e.g., faster) to work with right and equilateral triangles rather than scalene triangles. We opt for this approach. The length of the horizontal leg comes from the bottom-most right triangle.

$$y = \frac{x}{\tan(0.21\ \text{rad})} = 4.692\ x$$

Using the Pythagorean Theorem on the middle right triangle, we calculate the hypotenuse to be

$$c^2 = (2x)^2 + y^2 \quad \text{or} \quad c = \sqrt{4x^2 + (4.692x)^2} = 5.100\ x \ .$$

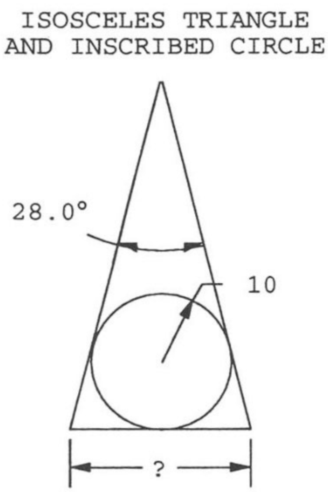
But the problem shows this value to be 1.86, so we use this information to solve for x.

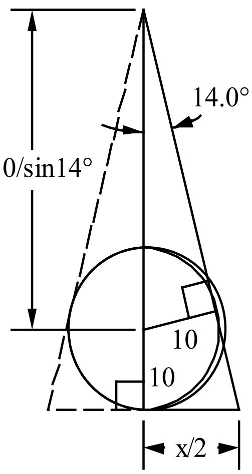
$$c = 5.100\,x = 1.86 \quad \text{or} \quad x = \frac{1.86}{5.100} = 0.3647$$

$$\text{and} \quad y = 4.692\,x = 1.711$$

The length of AB is the answer we seek. At this point, we could work on the upper-most scalene triangle. If we did, we would see that we know two lengths, c and 2x, as well as the obtuse angle calculated as the supplement of the angle in the middle right triangle. The Law of Cosines would be used to calculate AB. This is the hard way. Notice that the middle and upper-most triangles combine into a larger right triangle of leg dimensions y = 1.711 and 4x = 1.459. The hypotenuse is AB. The Pythagorean Theorem quickly provides us with the answer.

$$AB = \sqrt{(1.711)^2 + (1.459)^2} = 2.25$$

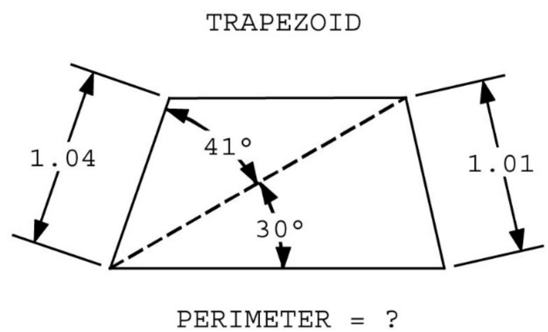




Geometry Example 4: Isosceles Triangle and Inscribed Circle (Left)

This problem may look formidable at first, but the trick is to cut the isosceles triangle in two such that we form two right triangles. This is often a rapid way to get a solution when dealing with isosceles triangles. Looking at the figure at right, two right triangles are formed, both sharing an acute angle of 28°/2 = 14°. The smaller one has a hypotenuse equal to 10/sin14° = 41.34. Adding this to the radius (10), we get the length of the longer, vertical leg of the larger right triangle, or 51.34. Keeping the larger right triangle in mind, the short leg is x/2 where x is the answer we seek. But it is also equal to the longer leg times tan14°. The answer x is twice this value or 2(51.34)tan14° = 25.6.

Geometry Example 5: Trapezoid. The trapezoid below is already split along its diagonal to form two scalene triangles. The perimeter is desired, so we need to calculate the lengths of both bases. The upper one comes fairly quickly. From geometry, we can identify on the upper-left scalene triangle two angles, 41° and 30°, and one length 1.04. The 30° angle comes from the equality of the angles formed when a line intersects two parallel lines. The top "base" b_t comes from the Law of Sines.



$$\frac{\sin A}{a} = \frac{\sin B}{b_t}$$

$$\frac{\sin 30°}{1.04} = \frac{\sin 41°}{b_t} \quad \text{or} \quad b_t = \frac{\sin 41°}{\sin 30°}(1.04) = 1.365$$

The bottom base b_b is the sum of three lengths shown (x, b_t, y). Two come from construction of right triangles by dropping vertical lines from the top base to the bottom base. From the left-most of these right triangles, the leg dimension x is

$$x = 1.04 \cos 71° = 0.3386 \quad .$$

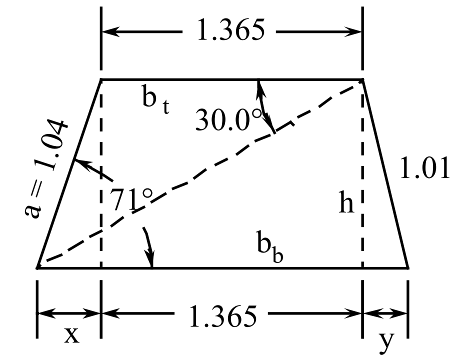
The altitude of the trapezoid h is the other leg of this triangle.

$$h = 1.04 \sin 71° = 0.9833$$

From the Pythagorean Theorem, the right-most right triangle has a shorter leg y equal to

$$y = \sqrt{1.01^2 - h^2} = \sqrt{1.01^2 - 0.9833^2} = 0.2305 \ .$$

The length of the bottom base b_b is the sum of x, y and the top base length b_t, or $b_b = 0.3386+0.2305+1.365 = 1.934$. The perimeter is$1.04+1.01+b_t+b_b = 5.35$.



E. Solving Geometry Problems on the Contest

Page 1, Problems 9 and 10. These are simple, usually one-step solutions for simple geometric figures like circles and rectangles. The companion Drill Manual for Stated and Geometry Problems lists example problems by geometric form with solutions for practice. Appendix B of this manual gives all the necessary formulas which you must know to successfully attack any geometry problem on the contest.

Page 2, Problems 19 and 20. These problems are one-step solutions to right triangles. They involve manipulation of a trig function like sine, cosine and tangent, as well as use of the area formula and the Pythagorean Theorem. Section B above describes the trig relations, and the Drill Manual for Stated and Geometry Problems provides practice problems.

Page 3, Problems 29 and 30. Simple solutions of solid (or three dimensional) geometric shapes are the subject of these problems. They are generally spheres, hemispheres, cylinders, cones, pyramids and frustums. The problems deal with length, area and volume relations as well as angular measure. Appendix B of this manual gives all the necessary formulas which you must know to solve any solid geometry problem on the contest. The Drill Manual for Stated and Geometry Problems gives example problems by geometric shape. Do not confuse the simple, Page 3 problems with the more complicated solid geometry problems that appear on Page 5.

Page 4, Problems 39 and 40. Problem 39 deals with circumscribed and inscribed circles on triangles. The Drill Manual also gives sample problems of this type. Appendix B lists the formulas for circles inside and around triangles categorized by triangle type: equilateral, right and scalene. Problem 40 exercises knowledge of the Law of Sines and the Law of Cosines. These relations are fully described in Section C of this chapter. Practice problems are listed in the Drill Manual for Stated and Geometry Problems.

Page 5, Problems 49 and 50. These problems are more complicated solid geometry problems. Generally, they are combinations of the simple geometric shapes dealt with on Page 3 problems. Fundamentally, the approach to a solution is the same as for Page 3 problems. The distinction is that multiple formulas are needed for the various shapes. The Drill Manual for Stated and Geometry Problems lists example problems.

Page 6, Problems 59 and 60. Problem 59 is a calculus problem. The two general types of problems are areas under curves and solids of revolution. Descriptions of the solution approaches are given in Chapter 4Lii and 4Liv of this manual, respectively. For solids of revolution, the area is swept around the y axis as shown in the last drawing of Figure 1. The Drill Manual for Stated and Geometry Problems lists quite a few example problems for practice. Problem 60 is a multiple step plane geometry problem generally dealing with length, area and angular relationships. They are complicated versions of Page 1, Page 2 and Page 4 problems. The solution is found by analyzing what is asked for, what given information is provided and how the given information can be used to find other intermediate unknowns that facilitate a solution. The formulas that form the basis for these problems are listed in Appendix B of this manual.

Page 7, Problems 64 and 65. These problems are generally complicated plane geometry problems typical of Problem 60 on Page 6.

Appendix A: Conversion Factors and Their Common Abbreviations

1 minute (min) = 60 seconds (s)
1 hour (hr) = 60 minutes (min)
1 day = 24 hours (hr)
1 week = 7 days
1 year = 12 months
1 year ≈ 365.256 days
1 century = 100 years
1 inch (in) = 2.54 centimeters (cm)
1 foot (ft) = 12 inches (in. or in)
1 yard (yd) = 3 feet (ft)
1 mile (mi) = 5,280 feet (ft)
1 sq. mile (sq. mi, mi^2) = 640 acres
1 nickel = 5 cents (pennies, ¢)
1 dime = 10 cents (¢)
1 quarter = 25 cents (¢)
1 half dollar = 50 cents (¢)
1 dollar ($) = 100 cents (¢)
1 silver dollar = 100 cents (¢)

Fluid measure, based on volume
1 teaspoon (tsp) = 1/6 ounce
1 tablespoon (tbs) = 0.5 ounce
1 cup = 8 ounces
1 pint (pt) = 16 ounces
1 quart (qt) = 32 ounces
1 half gallon = 64 ounces
1 gallon (gal) = 128 ounces
1 cubic foot (ft^3) ≈ 7.481 gallons (gal)
1 liter (l) ≈ 1.0567 quarts (qt)
1 pound (avdp, lb) = 16 ounces (avdp, oz)
1 pound (lb) ≈ 453.592 grams (g)
1 ton = 2000 pounds (lb)
1 revolution (rev) = 360 degrees (deg, °)
π radians (rad) = 180 degrees (deg, °)
1 degree (deg, °) = 60 minutes (') [angle measure]
1 minute (') = 60 seconds (") [angle measure]
°C = 5(°F - 32)/9 ≈ K - 273.15
where °C = Degrees Centigrade/Celsius
°F = Degrees Fahrenheit
K = Kelvins

Metric System prefixes:
nano (n, 10^{-9}), micro (μ, 10^{-6}), milli (m, 10^{-3}), centi (c, 10^{-2}), deci (d, 10^{-1}),
hecto (h, 100), kilo (k, 10^3), mega (M, 10^6), giga (G, 10^9), tera (T, 10^{12})

January has 31 days	February has 28 days*	March has 31 days
April has 30 days	May has 31 days	June has 30 days
July has 31 days	August has 31 days	September has 30 days
October has 31 days	November has 30 days	December has 31 days

*February has 29 days on leap years (leap years are years divisible by four and not divisible by 100)
Length of a football field (without end zones) = 100 yards (yd)
A card deck has 52 cards, 13 each of spades, hearts, diamonds and clubs. Sometimes 2 jokers are added.
Density (water) = 1 gram/cubic centimeter (g/cc, g/cm^3)
g = acceleration on earth ≈ -32.174 ft/s^2
Radius of the Earth ≈ 3960 mi
$1 + x + x^2 + x^3 + x^4 + x^5 + \ldots = (1 - x)^{-1} \quad (x^2<1)$

Appendix B: Geometry Formulas

Any reference book may be used to obtain the various formulas which contestants must know by memory to become proficient in solving geometric problems on the contest. As a convenience to coaches and contestants, we have assembled the formulas which we believe to be sufficient to solve geometric problems on the contest. Unless otherwise noted, angle measurement is given in degrees in the formulas, but on the contest both degree and radian measure are used.

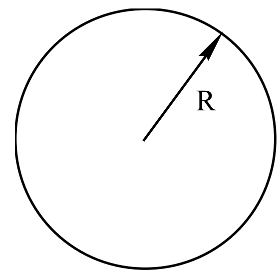
CIRCLE

R = radius, D = diameter, C = circumference, K = area

D = 2R

C = 2πR = πD

$$K = \pi R^2 = \frac{\pi}{4} D^2 = \frac{C^2}{4\pi}$$



SEGMENT AND SECTOR

R – radius, K = area, c = chord

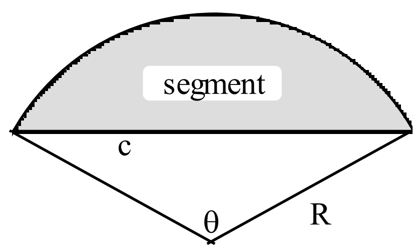
$$c = 2R\sin\left(\frac{\theta}{2}\right)$$

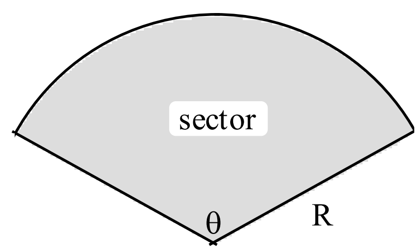
K(segment) = $\frac{1}{2}R^2(\theta - \sin\theta)$

[θ must be in radian measure]

K(sector) = $\frac{1}{2}R^2\theta$

[θ must be in radian measure]



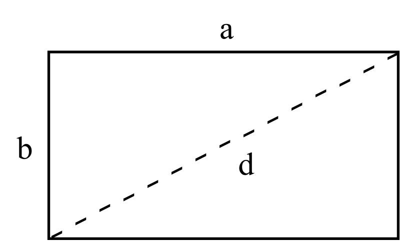


SQUARE AND RECTANGLE

A, b = side dimensions, p = perimeter, d = diagonal, K = area

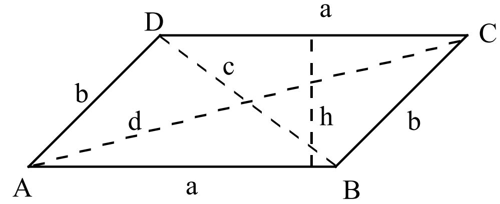
All internal angles = 90°

RECTANGLE	SQUARE
a≠b	a=b
$d = \sqrt{a^2 + b^2}$	$d = a\sqrt{2}$
p = 2(a+b)	p = 4a
K = ab	K = a^2



RHOMBUS AND PARALLELOGRAM

a, b = side dimension
c, d = diagonals, h = altitude on Side a
K = area, A, B, C, D = interior angles



PARALLELOGRAM

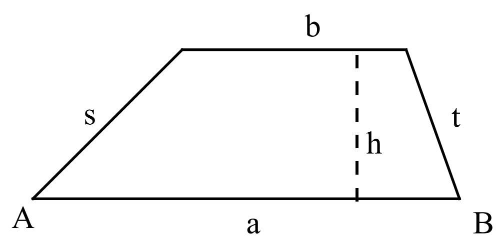
A = C, B = D, a ≠ b
A + B = 180°
$c = \sqrt{a^2 + b^2 - 2abcosA} = \sqrt{a^2 + b^2 + 2abcosB}$
$d = \sqrt{a^2 + b^2 + 2abcosA} = \sqrt{a^2 + b^2 - 2abcosB}$
p = 2(a+b)
h = b sinA = b sinB
K = ah = ab sinA = ab sinB

RHOMBUS

A = C, B = D, a = b
A + B = 180°
$c = a\sqrt{2(1 - cosA)}$
$= a\sqrt{2(1 + cosB)}$
$d = a\sqrt{2(1 + cosA)}$
$= a\sqrt{2(1 - cosB)}$
p = 4a
h = b sinA = b sinB
K = ah = ab sinA = ab sinB

TRAPEZOID

a, b, s, t = side dimensions, a and b are parallel
h = altitude on Side a, K = area

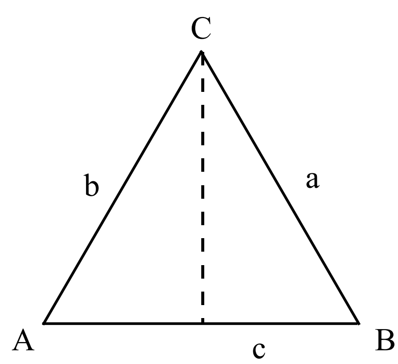


h = s sinA
h = t sinB

$$K = \frac{1}{2}(a + b)h$$

EQUILATERAL TRIANGLE

a = b = c = leg dimensions
A = B = C = 60°
h = altitude, K = area

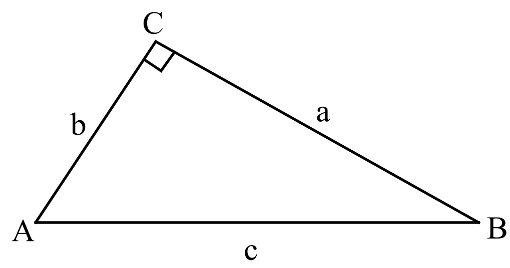


$$h = \frac{a\sqrt{3}}{2}$$

$$K = \frac{\sqrt{3}}{4}\, a^2$$

RIGHT TRIANGLE

a, b = leg dimensions, c = hypotenuse
A, B = acute angles, C = 90°, K = area



A + B + C = 180°

$c^2 = a^2 + b^2$ (Pythagorean Theorem)

K = $\frac{1}{2}$ ab

sin(A) = sinA = a/c

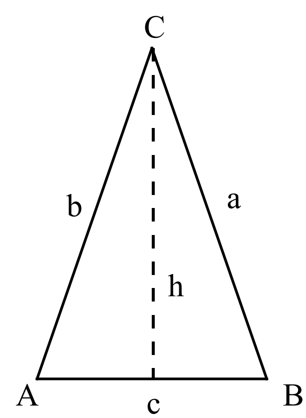
cos(A) = cosA = b/c

tan(A) = tanA = a/b

$[sin(A)]^2 + [cos(A)]^2 = sin^2 A + cos^2 A = 1$

ISOSCELES TRIANGLE

a, b, c = leg dimensions
A, B, C = angles
h = altitude on Side c
K = area



A = B   a = b   2A + C = 180°

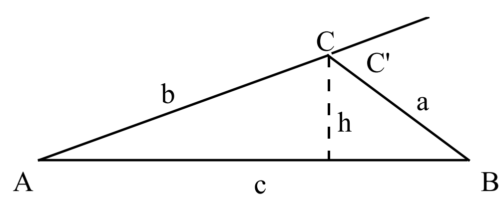
h = a sinA = $\frac{c}{2}$ tanA

$c^2 = 4(a^2 - h^2) = 4(1 - sin^2 A)a^2$

$K = \frac{1}{2} ch = \frac{1}{4} c^2 \, tanA$

SCALENE TRIANGLE

a, b, c = leg dimensions, h = altitude on Side C
A, B, C, C' = angles, K = area



A + B + C = 180°

h = a sinB = b sinA

K = $\frac{1}{2}$ bc sinA

$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c} = \frac{sinC'}{c}$ (Law of Sines)

$c^2 = a^2 + b^2 - 2ab \, cosC$ (Law of Cosines)

$s = \frac{1}{2}(a + b + c)$

$K = area = \sqrt{s(s-a)(s-b)(s-c)}$ (Heron's Formula)

CIRCUMSCRIBED AND INSCRIBED CIRCLES

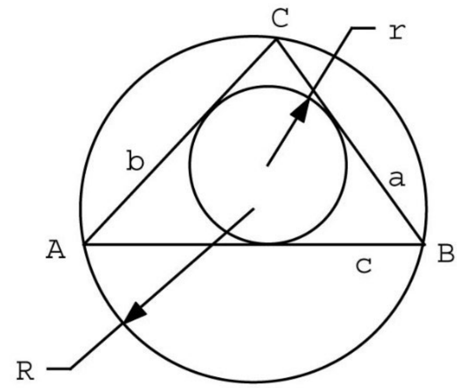
General (Scalene Triangle)

$$s = \frac{1}{2}(a+b+c)$$

$$r = (s-c)tan\left(\frac{C}{2}\right) = \frac{(a+b-c)}{2}\ tan\left(\frac{C}{2}\right)$$

$$R = \frac{abc}{4K} = \frac{a}{2\ sinA}$$ K = triangle area

$$r = 4R\ sin\left(\frac{A}{2}\right)\ sin\left(\frac{B}{2}\right)\ sin\left(\frac{C}{2}\right)$$



Right Triangle (c = hypotenuse)

$$r = \frac{a+b-c}{2}$$

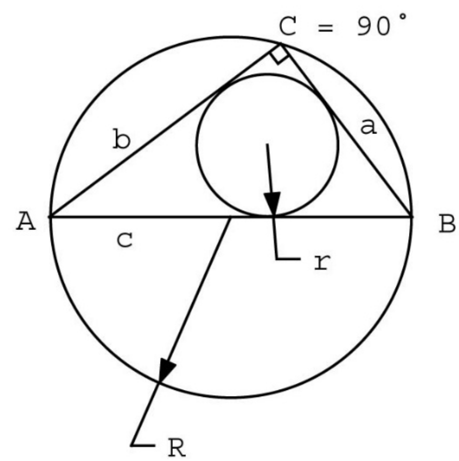
$$R = \frac{c}{2}$$

$$r = R\left(\frac{a+b-c}{c}\right)$$

Given r, c: a,b = $\frac{2r+c}{2} \pm \frac{\sqrt{8c^2-4(2r+c)^2}}{4}$

Given r, a: b = $\frac{2r(a-r)}{a-2r}$ and

c= (a – 2r) + $\frac{2r(a-r)}{a-2r}$ = a+b – 2r

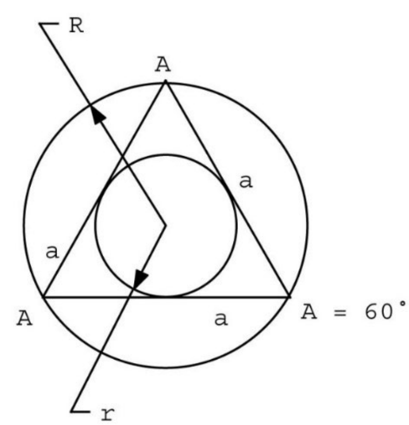


Equilateral Triangles

$$r = \frac{a\sqrt{3}}{6}$$

$$R = \frac{a\sqrt{3}}{3}$$

$$r = \frac{R}{2}$$

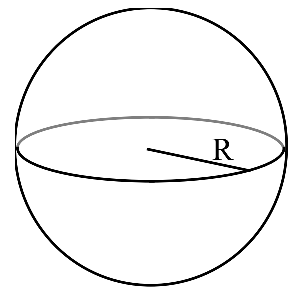


SPHERE

R = radius, D = diameter = 2R
S = surface area, V = volume

$$S = 4\pi R^2 = \pi D^2$$

$$V = \frac{4}{3}\pi R^3 = \frac{\pi}{6}D^3$$

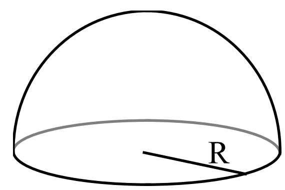


HEMISPHERE

R = radius, D = diameter = 2R
S = spherical surface area, T = total surface area
V = volume

$$S = 2\pi R^2 = \frac{\pi}{2}D^2 \qquad T = 3\pi R^2 = \frac{3\pi}{4}D^2$$

$$V = \frac{2}{3}\pi R^3 = \frac{\pi}{12}D^3$$



CYLINDERS

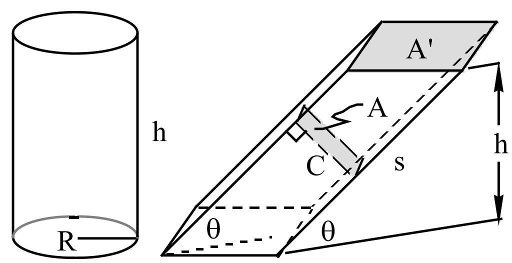
R = radius, h = altitude, s = slant height
C = right section perimeter, θ = slant angle
A = right section area, A′ = slant face area
S = lateral surface area, T = total surface area
V = volume

For right circular cylinders,

$S = 2\pi Rh \qquad T = 2\pi R(R+h) \qquad V = \pi R^2 h$

For slant or right cylinders of any cross-sectional shape,

$$S = sC = \frac{hC}{\sin\theta} \qquad A = A' \sin\theta \qquad V = hA' = sA$$

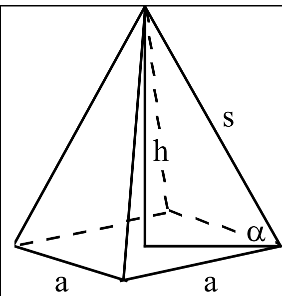


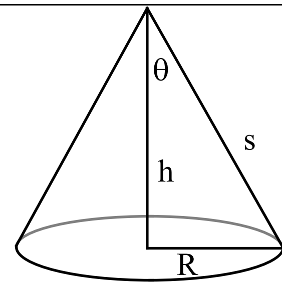
CONES AND PYRAMIDS

R = radius, a = side dimension
s = maximum slant height, h = altitude
θ = semicone angle
S = lateral surface area, T = total surface area
V = volume

For any right cone or pyramid,

$$V = \frac{h}{3}(area\ of\ the\ base)$$





SQUARE-BASE PYRAMID

$$h = \frac{a\ tan\alpha}{\sqrt{2}} \qquad s = \frac{h}{sin\alpha} = \frac{a}{\sqrt{2}\ cos\alpha}$$

$$S = \frac{a^2}{cos\alpha}\sqrt{1 + sin^2\alpha} \qquad T = S + a^2$$

$$V = \frac{a^2 h}{3}$$

RIGHT CIRCULAR CONE

$$s = \sqrt{R^2 + h^2} = \frac{h}{cos\theta} = \frac{R}{sin\theta}$$

S = πRs T = πR(R + s)

$$V = \frac{\pi}{3} R^2 h$$

FRUSTUM OF A RIGHT CIRCULAR CONE

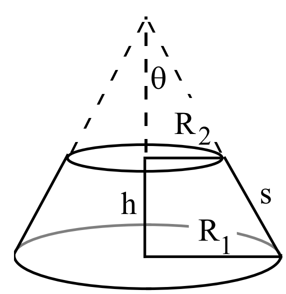
R1 = radius of lower base
R2 = radius of upper base
s = slant height, h = altitude
S = lateral surface area, T = total surface area
V = volume

$$s = \sqrt{(R_1 - R_2)^2 + h^2} = \frac{h}{cos\theta} = \frac{R_1 - R_2}{sin\theta}$$

$$S = \pi(R_1 + R_2)s$$

$$T = \pi[R_1^2 + R_2^2 + (R_1 + R_2)s]$$

$$V = \frac{\pi}{3}\ h\ (R_1^2 + R_2^2 + R_1 R_2) = \frac{\pi}{12}\ h\ (D_1^2 + D_2^2 + D_1 D_2)$$



Appendix C - Acceleration and Trajectory Equations
From Chapter 4Eii and 4Eiii

Acceleration Equations

General. a = constant acceleration, v = velocity, d = distance, t = time, v_o and d_o are associated values at which the acceleration initiates, and t_o is the time at which acceleration commences.

$$v = v_o + a(t - t_o) \qquad \text{and} \qquad d = d_o + v_o(t - t_o) + \frac{1}{2}a(t - t_o)^2$$

Specific. When t_o, v_o and d_o are all zero, the standard equations simplify to the more common forms:

$$v = at \quad \text{and} \quad d = \frac{1}{2}at^2 = \frac{1}{2}vt = \frac{1}{2}\frac{v^2}{a}$$

Trajectory Equations.

Initial and Final Elevations Equal. If v_o and θ are given, then the maximum horizontal and vertical distances are, respectively:

$$d_{h_{max}} = \frac{-v_o^2 \sin 2\theta}{g} \qquad \text{and} \qquad d_{v_{max}} = \frac{-v_o^2 \sin^2\theta}{2g}$$

$$d_{h_{max}} = \frac{-v_o^2\ sin(2\theta)}{g} \quad and \quad d_{v_{max}} = \frac{-v_o^2\ sin^2\theta}{2g} = \frac{d_v}{4\left[\left(\frac{d_h}{d_{h_{max}}}\right) - \left(\frac{d_h}{d_{h_{max}}}\right)^2\right]}$$

Given d_{hmax} and d_{vmax}, the required initial velocity v_o and angle θ are given by:

$$v_o = \sqrt{\left(\frac{-g}{8d_{v_{max}}}\right)\left(d_{h_{max}}^2 + 16d_{v_{max}}^2\right)} = \sqrt{-d_{h_{max}}g\left(\frac{1 + tan^2\theta}{2\ tan\theta}\right)} = \sqrt{\frac{d_h^2 g}{d_h\ sin(2\theta) - 2d_v\ cos^2\theta}}$$

$$tan\theta = \frac{4d_{v_{max}}}{d_{h_{max}}} = \frac{d_v/d_h}{1 - \frac{d_h}{d_{h_{max}}}}$$

The time of flight t_{of} is given by:

$$t_{of} = \frac{-2v_o \sin\theta}{g}$$

d_v given v_o, θ, and d_h:

$$d_v = d_h\ tan\theta + \frac{g d_h^2}{2v_o^2\ cos^2\theta}$$

Initial and Final Elevations Unequal. The starting elevation is d_{vo}, the final elevation is d_{vf}. If t_o is set equal to zero, any horizontal distance d_h can be written as a function of time:

$$d_h = v_o t \cos\theta \quad \text{or} \quad t = \frac{d_h}{v_o \cos\theta} \quad \text{and} \quad t_{of} = \frac{d_{h_{max}}}{v_o \cos\theta}$$

Any vertical distance d_v can likewise be written as

$$d_v = d_{vo} + v_o t \sin\theta + \frac{1}{2} g t^2 .$$

Setting this equal to the final vertical elevation and substituting the time relationship for d_h,

$$d_{vf} = d_{vo} + d_{h\,max} \tan\theta + \frac{g d_{h\,max}^2}{2 v_o^2 \cos^2\theta}$$

Appendix D – Compound Interest and Exponential Growth/Decay
From Chapter 4Giii

Simple Interest. Principal P_o invested for n periods at an interest rate i (per period) yields after n periods an amount P according to

$$P = P_o(1 + i)^n$$

Compound Interest. Principal P_o invested for n periods at an interest rate i (per period), compounded q times per period, yields after n periods an amount P according to

$$P = P_o(1 + i/q)^{mq}$$

Exponential Growth/Decay. A general growth or decay relation may be written relative to an arbitrary starting value P_o which grows or decays to a corresponding final value P after a time period t, where C and τ are constants:

$$P = P_o C^{t/\tau}$$

When C is greater than one, exponential growth occurs. When C is less than one, exponential decay occurs.

If C = 1/2, a radioactive decay function is obtained for which τ is defined to be the half-life:

$$P = P_o \left(\frac{1}{2}\right)^{t/\tau} = P_o (2)^{-t/\tau}$$

Judicious selection of C and τ facilitates problem solving. For example, if something triples every 7 hr, then the growth is described as

$$P = P_o (3)^{t/7\,hr}$$

Appendix E - Percent Problem Summary
From Chapter 4Gv
(Percent problems are noted by a "%" in the answer blank)

The first form of the equation better demonstrates the concept of a percent problem than the second form, but use of the second form is strongly recommended. It requires fewer key strokes. When working significant-digit stated problems, errors in the significant-digit part of the answer sometimes arise when numbers are repeated in the calculation. This is discussed in greater detail in the section on significant-digit stated problems.

What is the *percent change* if A changes to B?

$$\text{Percent Change} = 100\left[\frac{B-A}{A}\right] = 100\left[\frac{B}{A}-1\right]$$

Here, A is the "starting" number, the number for which a change is initiated.

Given an exact number E and approximate number A, what is the *percent error*?

$$\text{Percent Error} = 100\left[\frac{\text{A}-\text{E}}{\text{E}}\right] = 100\left[\frac{\text{A}}{\text{E}} - 1\right] .$$

Given a small number S and a large number L, what is the *percent increase*?

$$\text{Percent Increase} = 100\left[\frac{L-S}{S}\right] = 100\left[\frac{L}{S}-1\right] \quad \text{(answer always positive)}$$

Given a small number S and a large number L, what is the *percent decrease*?

$$\text{Percent Decrease} = 100\left[\frac{L-S}{L}\right] = 100\left[1-\frac{S}{L}\right] \quad \text{(answer always positive)}$$

Appendix F – Scaling Equations
From Chapter 4I

Geometrically similar figures obey scaling principles. L and h are arbitrary length dimensions on geometrically similar figures, A is any area, V is any volume. C_i are constants of proportionality.

$L = C_1 h$

$A = C_2 h^2$

$V = C_3 h^3 = C_4 A^{3/2}$

For two geometrically similar figures, Figure 1 and Figure 2,

$$\frac{L_2}{L_1} = \frac{h_2}{h_1}$$

$$\frac{A_2}{A_1} = \left(\frac{L_2}{L_1}\right)^2$$

$$\frac{V_2}{V_1} = \left(\frac{L_2}{L_1}\right)^3 = \left(\frac{A_2}{A_1}\right)^{3/2}$$

Appendix G - Best Fit Lines
From Chapter 4J

Given N (x_i,y_i) data pairs, $\bar{y}$ is the average of the y_i values and $\bar{x}$ is the average of all the x_i values.

$$\bar{x} = \frac{\sum_i x_i}{N} \qquad \text{and} \qquad \bar{y} = \frac{\sum_i y_i}{N}$$

The best-fit straight line is the equation y = mx+b where:

$$m = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}$$

and

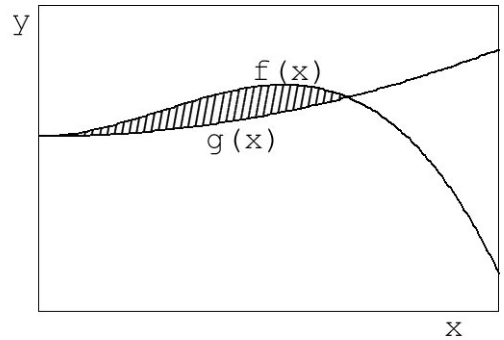
$$b = \bar{y} - m\bar{x}$$

The correlation coefficient r is a measure of how close the best fit line fits the data:

$$r = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2 \sum_i (y_i - \bar{y})^2}}$$

Rule for the Contest: The best fit line for y given x is NOT the same as the best fit line for x given y. For UIL problems, the data pairs as written in the problem statement define (x_i,y_i). Typically, the independent (often controllable) value is x, and the dependent (often measured) value is y. For example, if a golfer wanted to hit a ball 100 yd, but it actually went 120 yd, then the x value would be 100 yd, and the y value would be 120 yd.

Appendix H – Solids of Revolution
From Chapter 4Liv



<u>Rotating Parallel to the x-axis (y = b, *Disc Method*)</u>

$$V = \pi \int_{x_0}^{x_1} \left\{ [f(x) - b]^2 - [g(x) - b]^2 \right\} dx$$

<u>Rotating Parallel to the y-axis (x = a, *Shell Method*)</u>

$$V = 2\pi \int_{x_0}^{x_1} (x - a) \{ f(x) - g(x) \} dx$$

The axis of revolution for solids of revolution will be given in the title bar of the geometry problem, as shown in the examples on the reverse side of this page.

Appendix I – Calculus Derivatives and Integrals
From Chapter 4L

Summary Rules for Differentiation

u = f(x), v = g(x)   n, C = constants

$$\frac{dC}{dx} = 0$$

$$\frac{d(Cu)}{dx} = C\frac{du}{dx}$$

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d\left(u^n\right)}{dx} = nu^{n-1}\frac{du}{dx}$$

$$\frac{\mathrm{d}\left(\mathrm{C}^{\mathrm{u}}\right)}{\mathrm{dx}} = \mathrm{C}^{\mathrm{u}}\ \ln(\mathrm{C})\frac{\mathrm{du}}{\mathrm{dx}}$$

$$\frac{d(\sin u)}{dx} = \cos u\frac{du}{dx}$$

$$\frac{d(\cos u)}{dx} = -\sin u\frac{du}{dx}$$

$$\frac{d(\tan u)}{dx} = \sec^2 u\frac{du}{dx}$$

$$\frac{d\left(\sin^{-1} u\right)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d\left(\cos^{-1} u\right)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d\left(\tan^{-1} u\right)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

Summary Rules for Integration

u = f(x), v = g(x)   a, n, C, C' = constants

$$\int a\,du = a\int du$$
$$\int (du + dv) = \int du + \int dv$$
$$\int u\,dv = uv - \int v\,du$$
$$\int du = u + C$$
$$\int u^n du = \frac{u^{1+n}}{1+n} + C \qquad n \neq -1$$
$$\int \frac{du}{u} = \ln(u) + C$$
$$\int e^u du = e^u + C$$
$$\int a^u du = \frac{a^u}{\ln(a)} + C$$
$$\int \cos u\, du = \sin u + C$$
$$\int \sin u\, du = -\cos u + C$$
$$\int u\cos u\, du = \cos u + u\sin u + \mathrm{C}$$
$$\int u\sin u\, du = \sin u - u\cos u + \mathrm{C}$$
$$\int \sec^2 u\, du = \tan u + C$$
$$\int \frac{du}{\sqrt{1-a^2}} = \begin{cases} \sin^{-1} u + C \\ -\cos^{-1} u + C' \end{cases}$$
$$\int \frac{du}{1+u^2} = \tan^{-1} u + C$$

Note: All integrals with bounds x_1 and x_2 replace the constant of integration C with the difference of the integral at the bounded values, as in

$$\int_{x_1}^{x_2} du = u(x_2) - u(x_1)$$

Appendix J – Method of Least Significant Digits
From Chapter 4Miii

1. Numbers with limited accuracy are underlined in the problem statement, as in <u>5.06256</u>.

2. Trailing zeros in an underlined number are <u>not</u> significant if they do not cross the decimal point.

<u>57,200</u> has 3SD   <u>57200.00</u> has 7SD

3. Leading zeros in a number whose absolute value is between zero and one are not significant.

<u>0.00921</u> has 3SD   <u>4.00921</u> has 6SD

4. Significant-digit problems should be worked without any intermediate rounding. Only the final answer should be rounded after SD analysis of the solution.

5. Answers to significant-digit problems must be written with at least 2SD accuracy.

6. Any non-underlined conversion factors are considered exact even if they are intrinsically approximations.

7. For all operations except addition and subtraction (multiplication, division, powers, roots, trigonometric functions, etc.), the result has the same number of significant digits as the number with fewest significant digits.

0.0458 (3SD)/94.67 (4SD) = 0.000484 (3SD)
sin(32.456º) = 0.53665 (both are 5SD)
$\sqrt{9.123} = 3.020$ (both 4SD)

8. For addition and subtraction, round the answer to fit the absolute accuracy of the least accurate number in the calculation.

7,931.266 (7SD) + 7,930.924477 (10SD) = 15,862.190 (8SD)
7,931.266 (7SD) - 7,930.924477 (10SD) = 0.342 (3SD)

The first number is accurate to the thousandth's place and the second one is more accurate, being written to the millionth's place. The accuracy of the least accurate number (the first one) is to the thousandth's place, and the result of addition or subtraction of these numbers is to the same level, the thousandth's place.

9. Never repeat a number in the calculation of SD's for the answer. For example, compute the SD of the answer for a percent error calculation as

$$\%\text{Error} = 100\left[\frac{A}{E} - 1\right] \quad \text{rather than as} \quad \%\text{Error} = 100\left[\frac{A - E}{E}\right]$$

Appendix K - Matrix Algebra

From Chapter 4K

Addition: If **C** = **A** + **B**, then $C_{ij} = A_{ij} + B_{ij}$

Subtraction: If **C** = **A** - **B**, then $C_{ij} = A_{ij} - B_{ij}$

Scalar Multiplication: If a is a constant, then if **C** = a**B**, then $C_{ij} = aB_{ij}$

Multiplication of 2-Row Matrices: If **C** = **AB**, $C_{ij} = \sum_{k=1}^{2} A_{ik} B_{kj}$, **AB**≠**BA**

Multiplication of 3-Row Matrices: If **C** = **AB**, $C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj}$, **AB**≠**BA**

Determinant of a 2x2 Matrix: $\mathrm{Det}\mathbf{B} = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = B_{11}B_{22} - B_{12}B_{21}$

Determinant of a 3x3 Matrix:

$$\mathrm{Det}\mathbf{B} = \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} = B_{11}B_{22}B_{33} + B_{12}B_{23}B_{31} + B_{21}B_{32}B_{13} - B_{11}B_{32}B_{23} - B_{22}B_{31}B_{13} - B_{33}B_{21}B_{12}$$

Appendix L - Trigonometric Identities

a, b = constants

$\cos(-a) = \cos a$

$\sin(-a) = -\sin a$

$\cos(\pi \pm a) = -\cos a$

$\sin(\pi \pm a) = \mp \sin a$

$$\cos\left(\frac{\pi}{2} \pm a\right) = \mp \sin a$$

$$\sin\left(\frac{\pi}{2} \pm a\right) = \cos a$$

$\cos(2a) = \cos^2 a - \sin^2 a$

$\sin(2a) = 2 \sin a \cos a$

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

$$\cos^2 a + \sin^2 a = 1 \quad \text{(Pythagorean Theorem)}$$

Appendix M
Answers to Significant-Digit Practice Set on Page 13

Number	Fixed Notation	Scientific Notation	Number	Fixed Notation	Scientific Notation
64.2528547	64.3	$6.43x10^{1}$	–2527.810821	–2530	$–2.53x10^{3}$
–0.004851202	–0.00485	$–4.85x10^{–3}$	0.000056401	0.0000564	$5.64x10^{–5}$
–28.33408672	–28.3	$–2.83x10^{1}$	–0.511232545	–0.511	$–5.11x10^{–1}$
–0.058566597	–0.0586	$–5.86x10^{–2}$	–5.558254255	–5.56	$–5.56x10^{0}$
–0.091154255	–0.0912	$–9.12x10^{–2}$	–0.000038651	–0.0000387	$–3.87x10^{–5}$
0.351614191	0.352	$3.52x10^{–1}$	–0.008498004	–0.0085	$–8.5x10^{–3}$
–0.000552416	–0.000552	$–5.52x10^{–4}$	–0.000208107	–0.000208	$–2.08x10^{–4}$
0.00099347	0.000993	$9.93x10^{–4}$	–872.8743858	–873	$–8.73x10^{2}$
0.00090227	0.000902	$9.02x10^{–4}$	5.243578055	5.24	$5.24x10^{0}$
0.003125191	0.00313	$3.13x10^{–3}$	–0.620209752	–0.62	$–6.2x10^{–1}$
–5.76415E–05	–0.0000576	$–5.76x10^{–5}$	0.000970094	0.00097	$9.7x10^{–4}$
–0.070929216	–0.0709	$–7.09x10^{–2}$	–0.000056258	–0.0000563	$–5.63x10^{–5}$
0.000942375	0.000942	$9.42x10^{–4}$	–8181.138157	–8180	$–8.18x10^{3}$
–2542.145292	–2540	$–2.54x10^{3}$	–0.589933534	–0.59	$–5.9x10^{–1}$
0.526521273	0.527	$5.27x10^{–1}$	–0.004832374	–0.00483	$–4.83x10^{–3}$
0.458663032	0.459	$4.59x10^{–1}$	0.006656252	0.00666	$6.66x10^{–3}$
–0.000733427	–0.000733	$–7.33x10^{–4}$	–3.354455156	–3.35	$–3.35x10^{0}$
–0.055170573	–0.0552	$–5.52x10^{–2}$	–0.000036167	–0.0000362	$–3.62x10^{–5}$
–0.000852177	–0.000852	$–8.52x10^{–4}$	–0.006600091	–0.0066	$–6.6x10^{–3}$
0.842572564	0.843	$8.43x10^{–1}$	–0.991647014	–0.992	$–9.92x10^{–1}$
–13.94580006	–13.9	$–1.39x10^{1}$	–0.000037683	–0.0000377	$–3.77x10^{–5}$
–0.04111643	–0.0411	$–4.11x10^{–2}$	0.924285204	0.924	$9.24x10^{–1}$
28.60622804	28.6	$2.86x10^{1}$	–0.578585854	–0.579	$–5.79x10^{–1}$
–408.4813209	–408	$–4.08x10^{2}$	–0.267678559	–0.268	$–2.68x10^{–1}$
–0.501230757	–0.501	$–5.01x10^{–1}$	0.000348063	0.000348	$3.48x10^{–4}$
0.000007568	0.00000757	$7.57x10^{–6}$	–0.098086458	–0.0981	$–9.81x10^{–2}$